The Fallacy of Large Numbers

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for a presentation at UT Austin

April 18, 2003

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Can a well-diversified portfolio offer significant performance?

- The traditional diversification argument suggests not
  - A diversified portfolio with $\beta = 1$ looks like the benchmark
  - Probability of significant under- or over-performance is small
- An alternative perspective models superior performance
  - Example: a diversified portfolio outperforms the index
  - Outperformance tends to a positive constant
- Reconciling the two results
- Numerical results

Assumptions about returns—pairs example

$$R_{n,i} = \begin{cases} R^f + \beta_n(R^B - R^f) + s_n + \varepsilon_{n,1} & i = 1 \\ R^f + \beta_n(R^B - R^f) - s_n + \varepsilon_{n,2} & i = 2 \end{cases}$$

$$s_n = \begin{cases} \delta & \text{with probability } 1/2 \\ -\delta & \text{with probability } 1/2 \end{cases}$$

$$\frac{1}{N}\sum_{n=1}^{N} \beta_n = 1$$

- R^f — riskfree rate
- β_n — beta coefficient (market risk sensitivity)
- R^B — benchmark return
- s_n — signal known by the informed manager
- $\varepsilon_{n,i}$ — unknowable part of idiosyncratic risk

Return of a fixed portfolio

Put proportion $1/N$ in one stock (say $I(n)$) of each of the pairs of stocks

$$
\begin{aligned}
R^{fixed} &= \sum_{n=1}^{N} \frac{1}{N} R_{n,I(n)} \\
&= \sum_{n=1}^{N} \frac{1}{N} (R^f + \beta_n (R^B - R^f) + s_n^* + \varepsilon_{n,I(n)}) \\
&= \left(\frac{1}{N} \sum_{n=1}^{N} R^f \right) + \left(\frac{1}{N} \sum_{n=1}^{N} \beta_n \right) (R^B - R^f) + \left(\sum_{n=1}^{N} \frac{1}{N} (s_n^* + \varepsilon_{n,I(n)}) \right) \\
&= R^f + (R^B - R^f) + \frac{1}{N} \sum_{n=1}^{N} \left(s_n^* + \varepsilon_{n,I(n)} \right) \\
&= R^B + \frac{1}{N} \sum_{n=1}^{N} (s_n^* + \varepsilon_{n,I(n)}).
\end{aligned}
$$

$$
s_n^* \equiv \begin{cases} s_n & I(n) = 1 \\ -s_n & I(n) = 2 \end{cases}
$$

Excess return of a fixed portfolio

$$
\begin{aligned}
E[R^{fixed} - R^B] &= E\left[\frac{1}{N}\sum_{n=1}^{N}(s_n^* + \varepsilon_{n,I(n)})\right] \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
\text{var}(R^{fixed} - R^B) &= E\left[\left(\frac{1}{N}\sum_{n=1}^{N}(s_n^* + \varepsilon_{n,I(n)}) - 0\right)^2\right] \\
&= \frac{1}{N}(\delta^2 + \sigma^2).
\end{aligned}
$$

For $x > 0$, the Chebyshev inequality tells me that

$$
\begin{aligned}
\text{prob}(R^{fixed} - R^B > x) &< \frac{\text{var}(R^{fixed} - R^B)}{x^2} \\
&= \frac{\delta^2 + \sigma^2}{Nx^2}.
\end{aligned}
$$

which tends to 0 as N increases.

Return of an active portfolio

Put a proportion $1/N$ in the security $\iota(n)$ in the nth pair with good news:

$$\iota(n) \equiv \begin{cases} 1 & \text{if } s_n = \delta \\ 2 & \text{if } s_n = -\delta \end{cases}$$

$$\begin{aligned}
R^{active} &= \sum_{n=1}^{N} \frac{1}{N} R_{n,\iota(n)} \\
&= \sum_{n=1}^{N} \frac{1}{N} (R^f + \beta_n (R^B - R^f) + \delta + \varepsilon_{n,\iota(n)}) \\
&= \left(\frac{1}{N} \sum_{n=1}^{N} R^f \right) + \left(\frac{1}{N} \sum_{n=1}^{N} \beta_n \right) (R^B - R^f) + \delta + \left(\sum_{n=1}^{N} \frac{1}{N} \varepsilon_{n,\iota(n)} \right) \\
&= R^f + (R^B - R^f) + \delta + \frac{1}{N} \sum_{n=1}^{N} \varepsilon_{n,\iota(n)} \\
&= R^B + \delta + \frac{1}{N} \sum_{n=1}^{N} \varepsilon_{n,\iota(n)}.
\end{aligned}$$

Excess return of an active portfolio

$$
\begin{aligned}
E[R^{active} - R^B] &= E\left[\delta + \frac{1}{N}\sum_{n=1}^{N} \varepsilon_{n,\iota(n)}\right] \\
&= \delta,
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{var}(R^{active} - R^B) &= E\left[\left(\delta + \frac{1}{N}\left(\sum_{n=1}^{N} \varepsilon_{n,\iota(n)}\right) - \delta\right)^2\right] \\
&= \frac{1}{N}\sigma^2.
\end{aligned}
$$

For $x > 0$, the Chebyshev inequality tells me that

$$
\begin{aligned}
\mathrm{prob}(R^{active} - R^B < \delta - x) &\leq \frac{\mathrm{var}(R^{active} - R^B - \delta)}{x^2} \\
&= \frac{\sigma^2}{Nx^2}
\end{aligned}
$$

which tends to 0 as N increases.

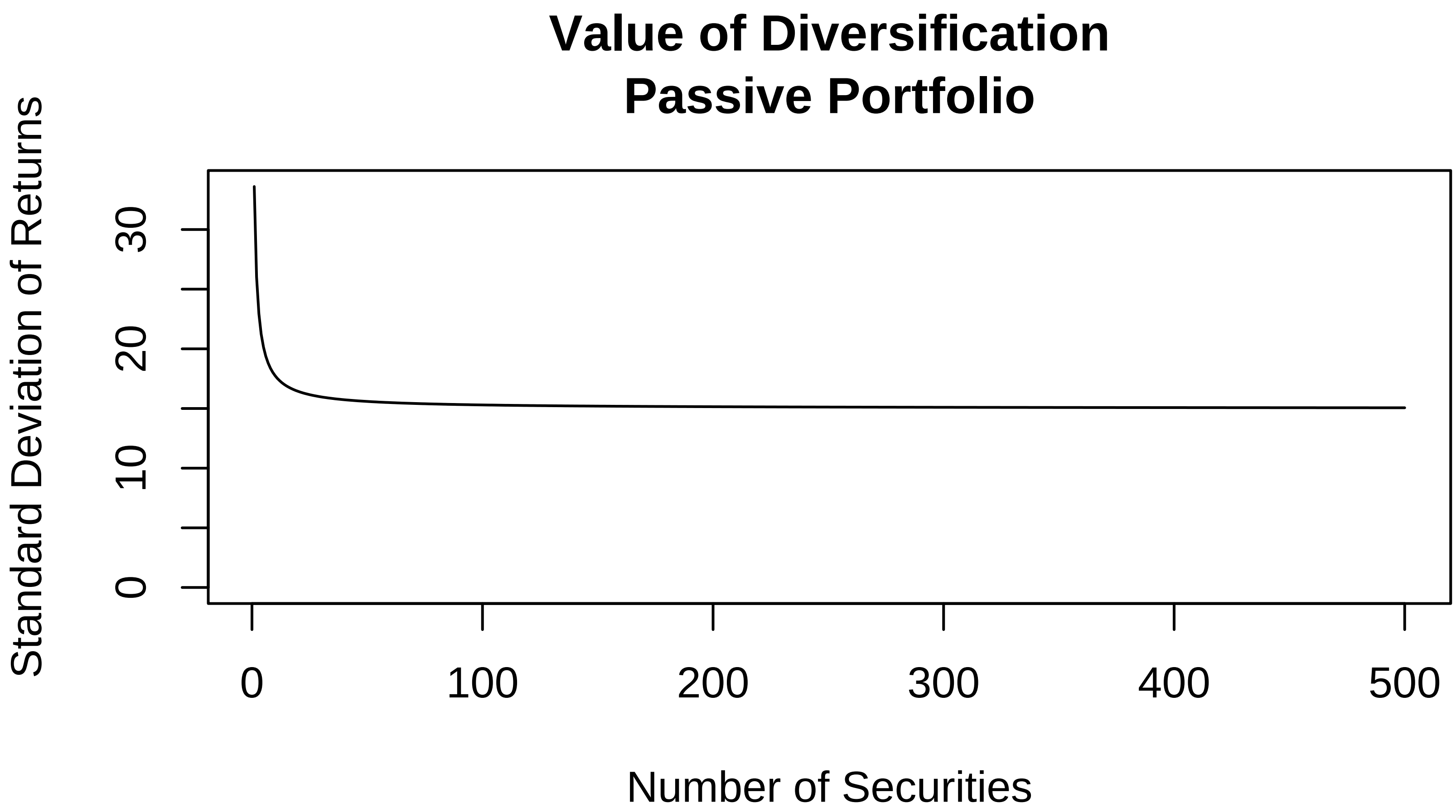
A paradox: how to reconcile the results

The portfolio that the manager holds given any realization of the signals is the same as a fixed portfolio that is known to be close to the index with very high probability. How then is it that the portfolio has an excess return close to δ with high probability?

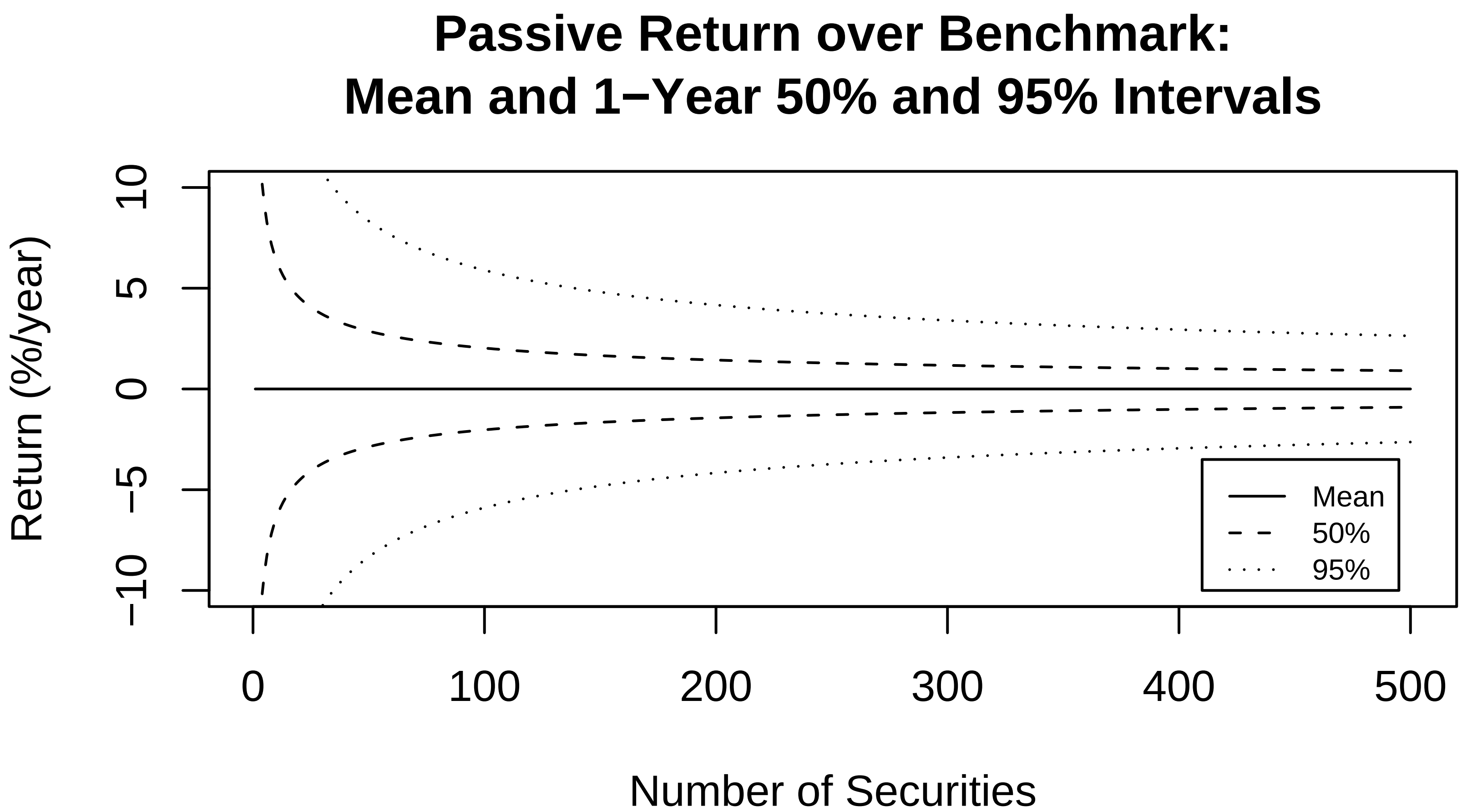
Indeed, the fixed portfolio has a return of close to δ with small probability. However, this particular portfolio is chosen by the active manager with *very* small probability $1/2^N$. Therefore, there is no conflict since we are only claiming that the fixed portfolio has a return close to δ a fraction $1/2^N$ of the time.

A case for large active portfolios

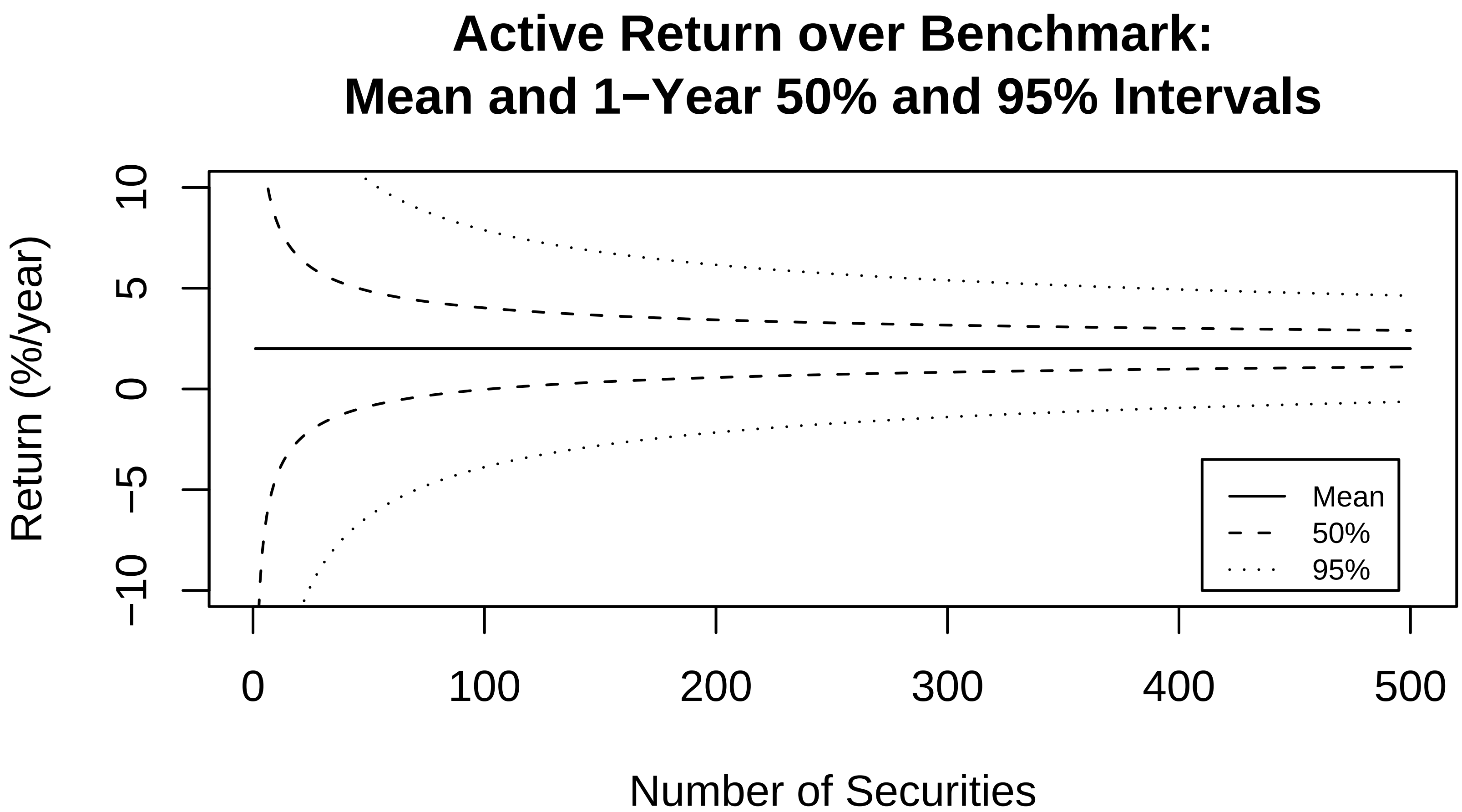
- Superior performance is not precluded
- Small idiosyncratic noise
- Ability easier to distinguish from luck in a well-diversified portfolio
- Less damage than for a small portfolio if performance is not realized



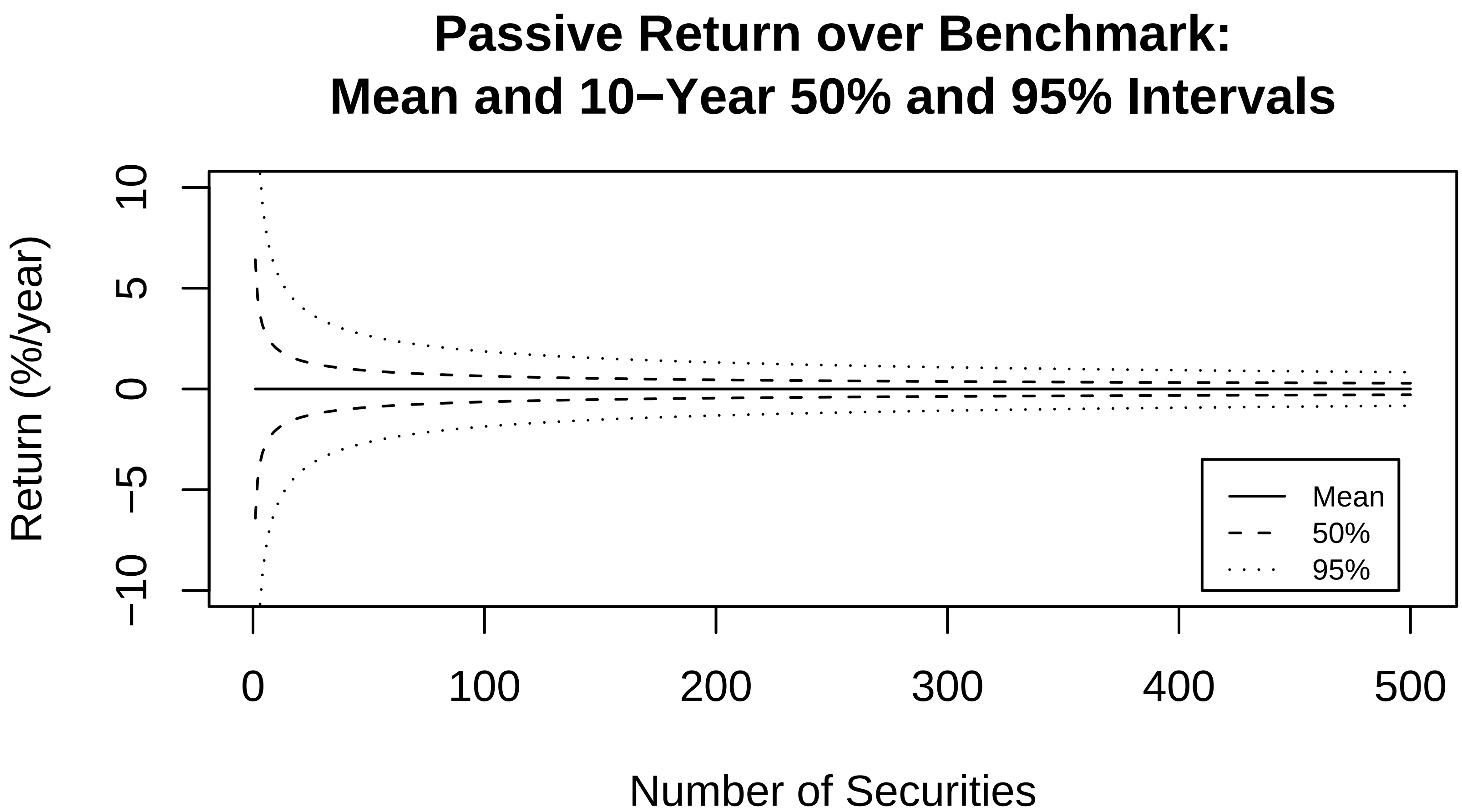
The value of diversification declines rapidly as we add securities, as is shown in this traditional graph.



As we add securities, the passive portfolio's deviation from the benchmark converges slowly to zero.



As we add securities, the active portfolio's deviation from the benchmark converges slowly to $+2\%$.



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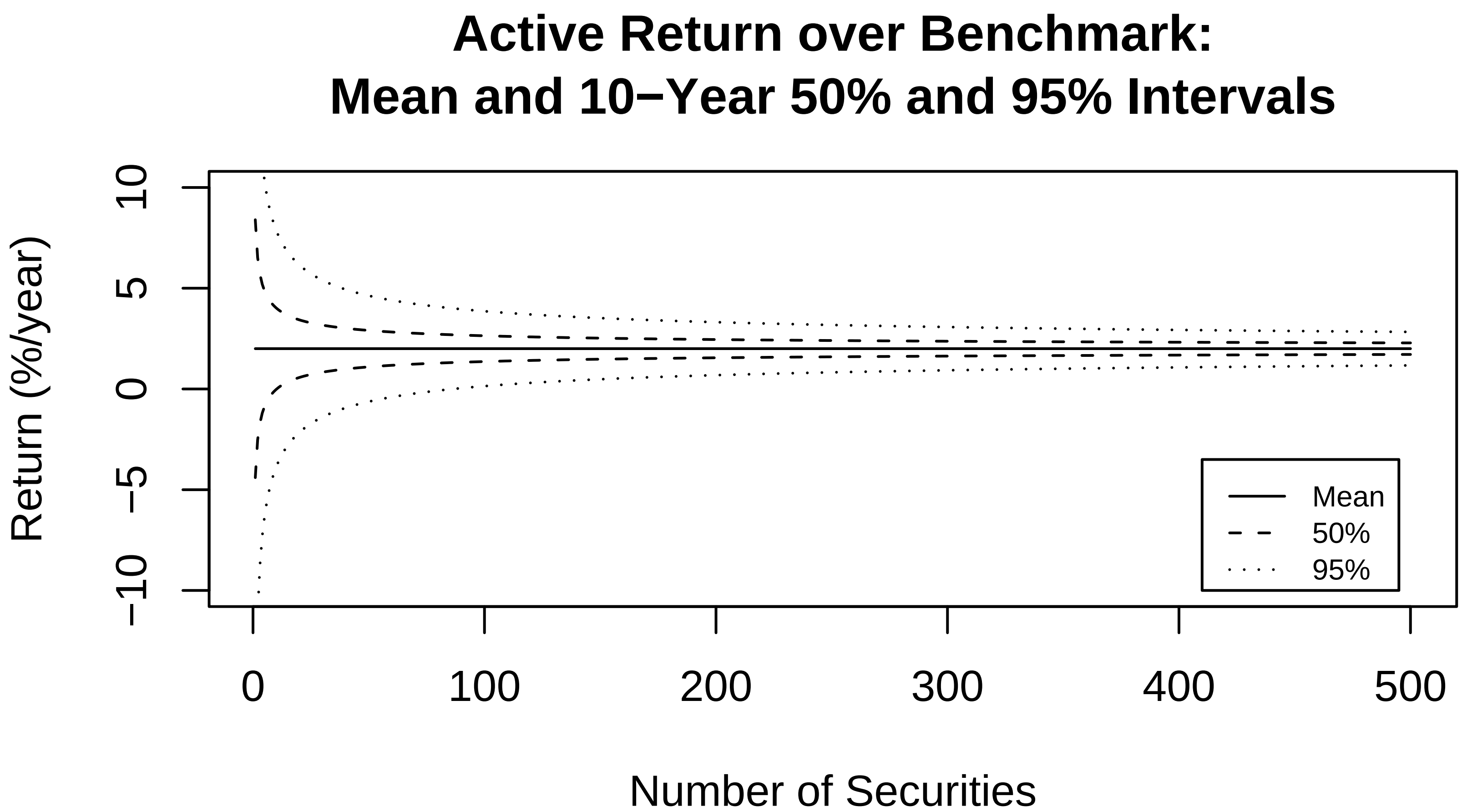

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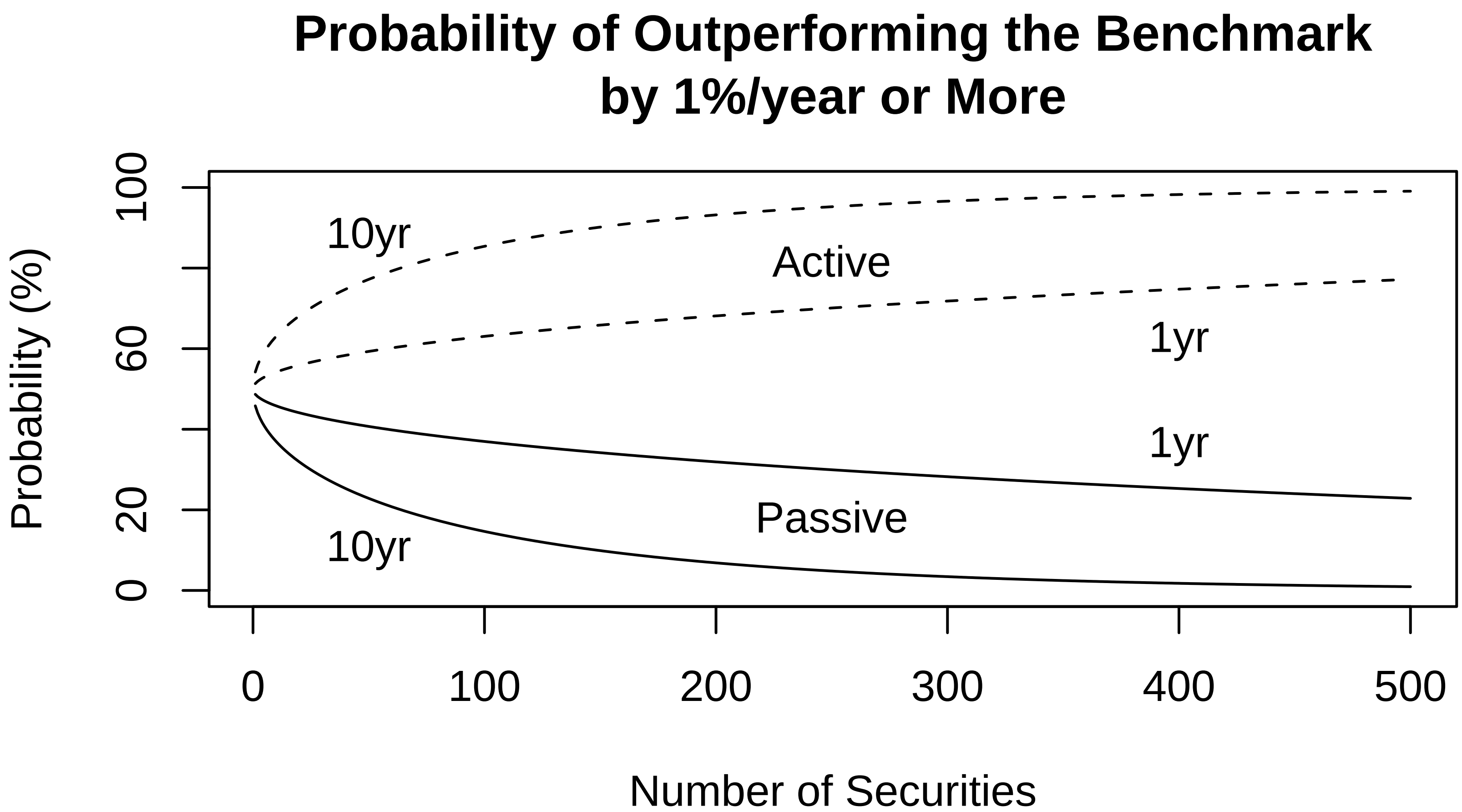

It becomes easier and easier to distinguish the active and passive portfolios as securities are added.

Summary

- Traditional analysis: large portfolios have returns close to the benchmark
- A new example shows that a large active portfolio can have superior performance
- This implies some possible advantages of large active portfolios
  - Superior performance is not precluded
  - Small idiosyncratic noise
  - Ability is easier to distinguish from luck in a well-diversified portfolio
  - Less damage than for a small portfolio if performance is not realized