

# Verification Theorems for Models of Optimal Consumption and Investment with Retirement and Constrained Borrowing<sup>\*</sup>

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## Abstract

We examine the impact of retirement flexibility and the constraint against borrowing from future labor income on optimal consumption, optimal investment, and optimal retirement policy. We solve three alternative models almost explicitly in parametric forms (up to at most a constant) and provide verification theorems for the claimed solutions. In addition, we also obtain analytical comparative statics.

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# I Introduction

Retirement is one of the most important economic events in a worker's life. This paper contains a rigorous formulation and analysis of several models of life-cycle consumption and investment, with voluntary or mandatory retirement, and with or without borrowing constraint against future labor income. In these models, optimal consumption jumps at retirement, and if retirement is voluntary, the optimal portfolio choice also jumps at retirement. If retirement is voluntary, the optimal retirement rule gives human capital a negative beta if wages are uncorrelated with the stock market, because retirement comes later when the market is down. This leads to aggressive investment in the market, a result that is dampened when borrowing against future labor income is prohibited and may be reversed when wages are positively correlated with market returns. More economic and qualitative description of the results can be found in the companion paper Dybvig and Liu (2009), while this paper focuses on the analytic results and rigorous proofs.

The main results in the paper are explicit parametric solutions (up to some constants) with rigorous verification theorems and analytical comparative statics. In particular, we combine the dual approach of Pliska (1986), He and Pagès (1993), and Karatzas and Wang (2000) with an analysis of the boundary to obtain a problem we can solve in a parametric form even if no known explicit solution exists in the primal problem. Having an explicit dual solution allows us to derive analytically the impact of parameter changes and, more importantly, allows us to prove a verification theorem showing that the first-order (Bellman equation) solution is a true solution to the choice problem. The proof is subtle because of (1) the nonconvexity introduced by the retirement decision, (2) the market incompleteness (from the agent's view) caused by the nonnegative wealth constraint, and (3) the technical problems caused by utility unbounded above or below. Two common approaches to proving a verification theorem are the dynamic programming (Fleming-Richel) approach and the separating hyperplane (Slater condition) approach. Both approaches encounter difficulties in our setting, so we use a hybrid of the two, using a separating hyperplane after retirement and dynamic programming before retirement. The two are combined with optional sampling: the continuation after retirement is replaced by the known value of the optimal continuation.

The rest of the paper is organized as follows. Section II presents the formal choice problems used in the paper. Section III presents the analytical solutions and proofs. Section IV closes the

paper.

## II Choice Problems

We consider the optimal consumption and investment problem of an investor who can continuously trade a risk free asset and  $n$  risky assets. The risk free asset pays a constant interest rate of  $r$ . The risky asset price vector  $S_t$  evolves as

$$\frac{dS_t}{S_t} = \mu dt + \sigma^\top dZ_t,$$

where  $Z_t$  is a standard  $n$  dimensional Wiener process,  $\mu$  ( $n \times 1$ ) and  $\sigma$  ( $n \times n$ ) are constants, and the division is element by element.

The investor also earns labor income  $y_t$ :

$$y_t \equiv y_0 \exp \left[ \left( \mu_y - \frac{\sigma_y^\top \sigma_y}{2} \right) t + \sigma_y^\top Z_t \right], \quad (1)$$

where  $y_0$  is the initial income from working,  $\mu_y$  and  $\sigma_y$  are constants of appropriate dimensions. The investor can choose to irreversibly retire at any point in time.

The investor has a constant mortality rate of  $\delta$ . The Poisson arrival time of mortality is denoted as  $\tau_d$  and is independent of the Wiener process  $Z_t$ . The investor can purchase insurance coverage of  $B_t - W_t$  against mortality, where  $W_t$  is the financial wealth of the investor at time  $t$ , so that if death occurs at  $t$ , the investor has a bequest of  $W_t + (B_t - W_t) = B_t$ . Insurance is assumed to be fairly priced at the mortality rate  $\delta$  per unit of coverage.

The investor derives utility from intertemporal consumption and bequest. The investor has a constant relative risk aversion (CRRA), time additive utility function (2) with a subjective time discount rate  $\rho$ :

$$E \left[ \int_{t=0}^{\tau_d} e^{-\rho t} \left( (1 - R_t) \frac{c_t^{1-\gamma}}{1-\gamma} + R_t \frac{(K c_t)^{1-\gamma}}{1-\gamma} \right) dt + e^{-\rho \tau_d} \frac{(k B_{\tau_d})^{1-\gamma}}{1-\gamma} \right], \quad (2)$$

where  $\gamma$  is the relative risk aversion coefficient, the constant  $K > 1$  indicates preferences for not working, in the sense that the marginal utility of consumption is greater after retirement than before

retirement, the constant  $k > 0$  measures the intensity of preference for leaving a large bequest, while the limit  $k^{1-\gamma} \rightarrow 0$  implements the special case with no preference for bequest, and  $R_t$  is the right-continuous and non-decreasing indicator of the retirement status at time  $t$ , which is 1 after retirement and 0 before retirement. The state variable  $R_{0-}$  is the retirement status at the beginning of the investment horizon.

Below are the three choice problems we focus on in this paper.

**Problem 1 (benchmark)** *Given initial wealth  $W_0$ , initial income from working  $y_0$ , and time-to-retirement  $T$  with associated retirement indicator function  $R_t = \iota(T \leq t)$ , choose adapted nonnegative consumption  $\{c_t\}$ , adapted portfolio  $\{\theta_t\}$ , and adapted nonnegative bequest  $\{B_t\}$ , to maximize expected utility of lifetime consumption and bequest*

$$E \left[ \int_{t=0}^{\tau_d} e^{-\rho t} \left( (1 - R_t) \frac{c_t^{1-\gamma}}{1-\gamma} + R_t \frac{(Kc_t)^{1-\gamma}}{1-\gamma} \right) dt + e^{-\rho \tau_d} \frac{(kB_{\tau_d})^{1-\gamma}}{1-\gamma} \right], \quad (3)$$

subject to the budget constraint

$$W_t = W_0 + \int_{s=0}^t (rW_s ds + \theta_s^\top ((\mu - r\mathbf{1}) ds + \sigma^\top dZ_s) + \delta(W_s - B_s) ds - c_s ds + (1 - R_s)y_s ds), \quad (4)$$

the labor income process (1) and the limited borrowing constraint

$$W_t \geq -g(t)y_t, \quad (5)$$

where

$$g(t) \equiv \begin{cases} \left( \frac{1 - e^{-\beta_1(T-t)}}{\beta_1} \right)^+ & \text{if } \beta_1 \neq 0 \\ (T-t)^+ & \text{if } \beta_1 = 0, \end{cases} \quad (6)$$

$$\beta_1 \equiv r + \delta - \mu_y + \sigma_y^\top \kappa \quad (7)$$

is the effective discount rate for labor income and assumed to be positive, and

$$\kappa \equiv (\sigma^\top)^{-1}(\mu - r\mathbf{1}) \quad (8)$$

is the price of risk.

**Problem 2 (VR)** Given initial wealth  $W_0$ , initial income from working  $y_0$ , and initial retirement status  $R_{0-}$ , choose adapted nonnegative consumption  $\{c_t\}$ , adapted portfolio  $\{\theta_t\}$ , adapted nonnegative bequest  $\{B_t\}$ , and adapted nondecreasing retirement indicator  $\{R_t\}$  (i.e., a right-continuous process taking values 0 and 1) to maximize the expected utility of lifetime consumption and bequest (2) subject to the budget constraint (4), the labor income process before retirement (1), and the limited borrowing constraint

$$W_t \geq -(1 - R_t) \frac{y_t}{\beta_1}. \quad (9)$$

**Problem 3 (VRNBC)** The same as Problem 2, except that the limited borrowing constraint is replaced by the no-borrowing constraint

$$W_t \geq 0. \quad (10)$$

To summarize the differences across the problems, moving from Problem 1 to Problem 2, the fixed retirement date  $T$  ( $R_t = \iota(t \geq T)$ ) is replaced by free choice of retirement date ( $R_t$  a choice variable), along with a technical change in the calculation of the maximum value of future labor income ((5) to (9)). Moving from Problem 2 to Problem 3 replaces the limited borrowing constraint (9) with a no-borrowing constraint (10).

### III The Analytical Solution and Comparative Statics

Let

$$\nu \equiv \frac{\gamma}{\rho + \delta - (1 - \gamma)(r + \delta + \frac{\kappa^\top \kappa}{2\gamma})}. \quad (11)$$

For our solutions, we will assume  $\nu > 0$ , which is also the condition for the corresponding Merton problem to have a solution, because if  $\nu < 0$ , then the investor can achieve infinite utility by delaying consumption.

**Theorem 1 (Benchmark)** Suppose  $\nu > 0$  and that the limited-borrowing constraint is satisfied with strict inequality at the initial values:

$$W_0 > -g(0)y_0. \quad (12)$$

Then in the solution to the investor's Problem 1, the optimal wealth process is

$$W_t^* = f(t)y_t x_t^{-1/\gamma} - g(t)y_t, \quad (13)$$

the optimal consumption policy is

$$c_t^* = K^{-bR_t} f(t)^{-1} (W_t^* + g(t)y_t), \quad (14)$$

the optimal trading strategy is

$$\theta_t^* = \frac{(\sigma^\top \sigma)^{-1}(\mu - r\mathbf{1})}{\gamma} (W_t^* + g(t)y_t) - \sigma^{-1} \sigma_y g(t)y_t, \quad (15)$$

and the optimal bequest policy is

$$B_t^* = k^{-b} f(t)^{-1} (W_t^* + g(t)y_t), \quad (16)$$

where

$$b \equiv 1 - 1/\gamma, \quad (17)$$

$$f(t) \equiv (\hat{\eta} - \eta) \exp\left(-\frac{1 + \delta k^{-b}}{\eta} (T - t)^+\right) + \eta, \quad (18)$$

$$\eta \equiv (1 + \delta k^{-b})\nu, \quad (19)$$

$$\hat{\eta} \equiv (K^{-b} + \delta k^{-b})\nu. \quad (20)$$

$$x_t \equiv \left( \frac{W_0 + g(0)y_0}{y_0 f(0)} \right)^{-\gamma} e^{(\mu_x - \frac{1}{2}\sigma_x^\top \sigma_x)t + \sigma_x^\top Z_t}, \quad (21)$$

$$\mu_x \equiv -(r - \rho) - \frac{1}{2}\gamma(1 - \gamma)\sigma_y^\top \sigma_y + \gamma\mu_y - \gamma\sigma_y^\top \kappa, \quad (22)$$

and

$$\sigma_x \equiv \gamma\sigma_y - \kappa. \quad (23)$$

Furthermore, the value function for the problem is

$$V(W, y, t) = f(t)^\gamma \frac{(W + g(t)y)^{1-\gamma}}{1-\gamma}.$$

The following lemmas are useful for proving Theorem 1.

**Lemma 1** *Suppose  $\nu > 0$ , and define  $\{R_s\}$  as in Problem 1. Then*

(i) *For any  $t$ ,  $g(t)y_t = \frac{1}{\xi_t} E_t[\int_t^\infty \xi_s(1 - R_s)y_s ds]$ . (In Problem 1,  $g(t)y_t$  is the market value at  $t$  of the subsequent labor income.)*

(ii) *For any strategy  $(c, \theta, B, W)$  that is feasible for Problem 1,*

$$E \left[ \int_0^\infty \xi_s(c_s + \delta B_s) ds \right] \leq W_0 + g(0)y_0, \quad (24)$$

*with equality for the claimed optimal strategy  $(c^*, \theta^*, B^*, W^*)$  in Theorem 1. (This is the static budget constraint. Inequality for a general policy may be due to investing wealth forever without consuming and/or a suicidal strategy.)*

PROOF OF LEMMA 1 (i) By Itô's lemma, (1), (6), (7), (36), and simple algebra,

$$d(\xi_t g(t)y_t) = -\xi_t(1 - R_t)y_t dt + \xi_t g(t)y_t(\sigma_y - \kappa)^\top dZ_t.$$

Furthermore,

$$E \int_0^t (\xi_s g(s)y_s)^2 (\sigma_y - \kappa)^\top (\sigma_y - \kappa) ds < \infty,$$

since  $\xi_s y_s$  is a standard lognormal diffusion and the other factors are bounded for any  $t$  and zero for  $t > T$ . Therefore, the local martingale

$$\xi_t g(t)y_t + \int_0^t \xi_s(1 - R_s)y_s ds = g(0)y_0 + \int_{s=0}^t \xi_s g(s)y_s(\sigma_y - \kappa)^\top dZ_s \quad (25)$$

is a martingale that is constant for  $t > T$ . Picking any  $\mathcal{T} > \max(t, T)$ , the definition of a martingale implies that

$$\xi_t g(t)y_t + \int_0^t \xi_s(1 - R_s)y_s ds = E_t \left[ \xi_{\mathcal{T}} g(\mathcal{T})y_{\mathcal{T}} + \int_0^{\mathcal{T}} \xi_s(1 - R_s)y_s ds \right] \quad (26)$$

Now,  $\mathcal{T} > \max(t, T)$  implies that  $g(\mathcal{T}) = 0$ , and the integral on the left-hand-side is known at  $t$ . Therefore, we can subtract the integral from both sides and divide both sides by  $\xi_t$  to conclude

$$\begin{aligned} g(t)y_t &= \frac{1}{\xi_t} E_t \left[ \int_t^{\mathcal{T}} \xi_s (1 - R_s) y_s ds \right] \\ &= \frac{1}{\xi_t} E_t \left[ \int_t^{\infty} \xi_s (1 - R_s) y_s ds \right], \end{aligned} \quad (27)$$

where the second equality follows from the fact that  $R_s \equiv 1$  for  $s > \mathcal{T}$ .

(ii) The limited borrowing constraint (5), together with the nonnegativity of  $c_t$ ,  $B_t$ , and  $\xi_t$ , imply that

$$\xi_t W_t + \int_0^t \xi_s (c_s + \delta B_s - (1 - R_s) y_s) ds \geq -[\xi_t g(t) y_t + \int_0^t \xi_s (1 - R_s) y_s ds]. \quad (28)$$

For any strategy  $(c, \theta, B, W)$  that is feasible in Problem 1, Itô's lemma, the budget constraint (4), the definition of  $g$ , (36), and (8) imply that the left-hand side of (28) has zero drift and is therefore a local martingale. Furthermore, the right-hand side of (28) is a martingale, which was an intermediate result that (25) is a martingale in the proof of Part (i) above. Since any local martingale bounded below by a martingale is a supermartingale, the left-hand-side of (28) is a supermartingale for any feasible strategy.

For the claimed optimal strategy, the left-hand side of (28) is a martingale, since it is a local martingale and the integrand with respect to  $dZ_t$  is the sum of lognormal terms that can be shown to be bounded in  $L^2$  over finite time intervals.

By definition of supermartingale and martingale, we have for any  $t > 0$  that

$$W_0 \geq E \left[ \xi_t W_t + \int_0^t \xi_s (c_s + \delta B_s - (1 - R_s) y_s) ds \right], \quad (29)$$

with equality for the claimed optimum. Taking the limit as  $t \uparrow \infty$  and using the result from (i), we have that

$$W_0 \geq \liminf_{t \uparrow \infty} E[\xi_t W_t] + E \left[ \int_0^{\infty} \xi_s (c_s + \delta B_s) ds \right] - g(0) y_0. \quad (30)$$

Since  $g(t) = 0$  for  $t > T$ , the no-borrowing-without-repayment constraint (5) implies that

$$\liminf_{t \uparrow \infty} E[\xi_t W_t] \geq 0$$

for any feasible strategy. Furthermore,  $g(t) = 0$  for  $t > T$  also implies that the claimed optimal wealth  $W_t^*$  is lognormal for  $t > T$  and it is straightforward to verify that  $\nu > 0$  implies that  $\lim_{t \uparrow \infty} E[\xi_t W_t^*] = 0$ . Therefore,

$$W_0 \geq E \left[ \int_0^\infty \xi_s (c_s + \delta B_s) ds \right] - g(0)y_0, \quad (31)$$

with equality for the claimed optimum, which can be rewritten as what is to be shown.  $\square$

**Lemma 2** (i) Let  $y_t, x_t, \xi_t$  be as defined in (1), (21), and (36) respectively. Then

$$\xi_t = e^{-(\rho+\delta)t} \left( \frac{y_t}{y_0} \right)^{-\gamma} \left( \frac{x_t}{x_0} \right). \quad (32)$$

(ii) Given any feasible strategy  $(c, B)$  for Problem 1 and the claimed optimal strategy  $(c^*, B^*)$  in Theorem 1, we have

$$\begin{aligned} & E \left[ \int_0^\infty e^{-(\rho+\delta)t} \left( \frac{(K^{R_t} c_t)^{1-\gamma}}{1-\gamma} + \delta \frac{(kB_t)^{1-\gamma}}{1-\gamma} \right) dt \right] \\ & \leq E \left[ \int_0^\infty e^{-(\rho+\delta)t} \left( \frac{(K^{R_t} c_t^*)^{1-\gamma}}{1-\gamma} + \delta \frac{(kB_t^*)^{1-\gamma}}{1-\gamma} \right) dt \right] \\ & \quad + \frac{x_0}{y_0^\gamma} E \left[ \int_0^\infty \xi_t (c_t + \delta B_t - c_t^* - \delta B_t^*) dt \right]. \end{aligned} \quad (33)$$

PROOF OF LEMMA 2: (i) This can be directly verified using (36), (21), (22), (23), and (1)

(ii) Because any concave function lies below the tangent line at any point, we have that for any positive  $c, c^*, B,$  and  $B^*$ , we have

$$\frac{(K^R c)^{1-\gamma}}{1-\gamma} \leq \frac{(K^R c^*)^{1-\gamma}}{1-\gamma} + (K^R)^{1-\gamma} c^{*\gamma} (c - c^*) \quad (34)$$

and

$$\frac{(kB)^{1-\gamma}}{1-\gamma} \leq \frac{(kB^*)^{1-\gamma}}{1-\gamma} + k^{1-\gamma} B^{*\gamma} (B - B^*). \quad (35)$$

Then (33) follows from (14), (16), and (32).  $\square$

PROOF OF THEOREM 1. The proof uses a separating hyperplane to separate preferred consumptions from the feasible consumptions. The feasibility of the claimed optimum follows from direct substitution. Given (12),  $W^*$  is well-defined by (13). It is tedious but straightforward to verify the budget equation (4) using Itô's lemma and the claimed form of the strategy  $(c^*, \theta^*, B^*, W^*)$  in (13)–(16), various definitions (6)–(8) and (13)–(23), and the definition of labor income (1). Note that  $W_0^* = W_0$  by (13) and (21). The no-borrowing-without-repayment constraint (5) follows from the positivity of  $f(t)$  and  $x$ , and the definition of  $W^*$  in (13).

We start by noting the state-price density and pricing results, both for labor income and for consumption and bequest. Define the state price density process  $\xi$  by

$$\xi_t \equiv e^{-(r+\delta+\frac{1}{2}\kappa^\top\kappa)t-\kappa^\top Z_t}. \quad (36)$$

This is the usual state-price density but adjusted to condition on living, given the mortality rate  $\delta$  and fair pricing of long and short positions in term life insurance.

As shown in Lemma 1 in the Appendix,  $g(t)y_t$  is the value at  $t$  of subsequent labor income, where  $g$  is defined in (6). (Note that  $g(t) \equiv 0$  for  $t \geq T$ .) Furthermore, by Lemma 1, we have that the present value of future consumption and bequest is no larger than initial wealth:

$$E\left[\int_0^\infty \xi_t(c_t + \delta B_t)dt\right] \leq W_0 + g(0)y_0, \quad (37)$$

for any feasible strategy, with equality for our claimed optimum.

We then have, after integrating out the mortality risk, for any feasible strategy  $(c, \theta, B)$ ,

$$\begin{aligned} & E\left[\int_0^\infty e^{-(\rho+\delta)t} \left( \frac{(K^{R_t}c_t)^{1-\gamma}}{1-\gamma} + \delta \frac{(kB_t)^{1-\gamma}}{1-\gamma} \right) dt\right] \\ & \leq E\left[\int_0^\infty e^{-(\rho+\delta)t} \left( \frac{(K^{R_t}c_t^*)^{1-\gamma}}{1-\gamma} + \delta \frac{(kB_t^*)^{1-\gamma}}{1-\gamma} \right) dt\right] \\ & \quad + \frac{x_0}{y_0} E\left[\int_0^\infty \xi_t(c_t + \delta B_t - c_t^* - \delta B_t^*)dt\right] \\ & \leq E\left[\int_0^\infty e^{-(\rho+\delta)t} \left( \frac{(K^{R_t}c_t^*)^{1-\gamma}}{1-\gamma} + \delta \frac{(kB_t^*)^{1-\gamma}}{1-\gamma} \right) dt\right], \end{aligned} \quad (38)$$

where the first inequality follows from Lemma 2 and the second inequality follows from pricing (37) for all strategies and equality for the claimed optimum. This says that the claimed optimum dominates all other feasible strategies. We showed previously that the claimed optimum is feasible, so it must indeed be optimal.  $\square$

Unlike Problem 1, Problems 2 and 3 do not seem to have explicit solutions in terms of the primal variables. However, we provide explicit solutions (up to at most one constant) in terms of marginal utility in Theorems 2 and 3. Recall the definition of  $\nu$  in (11) and  $\beta_1$  in (7), and define

$$\beta_2 \equiv \rho + \delta + \frac{1}{2}\gamma(1-\gamma)\sigma_y^\top \sigma_y - (1-\gamma)\mu_y \quad (39)$$

and

$$\beta_3 \equiv (\gamma\sigma_y - \kappa)^\top (\gamma\sigma_y - \kappa). \quad (40)$$

Then, here is the solution for the VR case.

**Theorem 2 (VR)** *Suppose  $\nu > 0$ ,  $\beta_1 > 0$ ,  $\beta_2 > 0$ , and that the no-borrowing-without-repayment constraint holds with strict inequality at the initial condition:*

$$W_0 > -(1 - R_{0-}) \frac{y_0}{\beta_1}. \quad (41)$$

*The solution to the investor's Problem 2 can be written in terms of the dual variable  $x_t$  (a normalized marginal utility of consumption). Specifically, let the dual variable be defined by*

$$x_t \equiv x_0 e^{(\mu_x - \frac{1}{2}\sigma_x^\top \sigma_x)t + \sigma_x^\top Z_t}, \quad (42)$$

where  $x_0$  solves

$$-y_0 \varphi_x(x_0, R_0) = W_0, \quad (43)$$

where

$$\varphi(x, R) = \begin{cases} -\hat{\eta} \frac{x^b}{b} & \text{if } R = 1 \text{ or } x \leq \underline{x} \\ A_+ x^{\alpha_-} - \hat{\eta} \frac{x^b}{b} + \frac{1}{\beta_1} x & \text{otherwise,} \end{cases} \quad (44)$$

where  $b$ ,  $\eta$ , and  $\hat{\eta}$  are as defined in Theorem 1 (in equations (17), (19), and (20)), and

$$A_+ \equiv \frac{1}{\gamma(b - \alpha_-)\beta_1} \underline{x}^{1-\alpha_-}, \quad (45)$$

the optimal retirement boundary is

$$\underline{x} = \left( \frac{(\eta - \hat{\eta})(b - \alpha_-)\beta_1}{b(1 - \alpha_-)} \right)^\gamma, \quad (46)$$

where

$$\alpha_- \equiv \frac{\beta_1 - \beta_2 + \frac{1}{2}\beta_3 - \sqrt{(\beta_1 - \beta_2 + \frac{1}{2}\beta_3)^2 + 2\beta_2\beta_3}}{\beta_3}. \quad (47)$$

Then the optimal consumption policy is

$$c_t^* = K^{-bR_t^*} y_t x_t^{-1/\gamma}, \quad (48)$$

the optimal trading strategy is

$$\theta_t^* = y_t [(\sigma^\top \sigma)^{-1}(\mu - r\mathbf{1})x_t \varphi_{xx}(x_t, R_t^*) - \sigma^{-1} \sigma_y (\gamma x_t \varphi_{xx}(x_t, R_t^*) + \varphi_x(x_t, R_t^*))], \quad (49)$$

the optimal bequest policy is

$$B_t^* = k^{-b} y_t x_t^{-1/\gamma}, \quad (50)$$

the optimal retirement policy is

$$R_t^* = \iota\{t \geq \tau^*\}, \quad (51)$$

the corresponding retirement wealth threshold is

$$\bar{W}_t = -y_t \varphi_x(\underline{x}, 0), \quad (52)$$

and the optimal wealth is

$$W_t^* = -y_t \varphi_x(x_t, R_t^*), \quad (53)$$

where

$$\tau^* = (1 - R_0) \inf\{t \geq 0 : x_t \leq \underline{x}\}. \quad (54)$$

Furthermore, the value function is

$$V(W, y, R) = y^{1-\gamma} (\varphi(x, R) - x\varphi_x(x, R)), \quad (55)$$

where  $x$  solves

$$-y\varphi_x(x, R) = W. \quad (56)$$

PROOF. See the proof after Theorem 3.

The following theorem provides an almost explicit solution to the VRNBC case with the no-borrowing constraint.

**Theorem 3 (VRNBC)**

Suppose  $\nu > 0$ ,  $\beta_1 > 0$ ,  $\beta_2 > 0$ , and that initial wealth is strictly positive:

$$W_0 > 0. \quad (57)$$

The solution to the investor's Problem 3 is similar to the solution to Problem 2, and can be written in terms of the new dual variable  $x_t$  defined by

$$x_t = \frac{x_0 e^{(\mu_x - \frac{1}{2}\sigma_x^\top \sigma_x)t - \sigma_x^\top Z_t}}{\max(1, \sup_{0 \leq s \leq \min(t, \tau^*)} x_0 e^{(\mu_x - \frac{1}{2}\sigma_x^\top \sigma_x)s - \sigma_x^\top Z_s} / \bar{x})} \quad (58)$$

where  $x_0$  solves

$$-y_0 \varphi_x(x_0, R_0) = W_0, \quad (59)$$

and  $\mu_x$  and  $\sigma_x$  are the same as in Theorem 2 (as given by (22) and (23)). The new dual value function is

$$\varphi(x, R) = \begin{cases} -\hat{\eta} \frac{x^b}{b} & \text{if } R = 1 \text{ or } x \leq \underline{x} \\ A_+ x^{\alpha_-} + A_- x^{\alpha_+} - \eta \frac{x^b}{b} + \frac{1}{\beta_1} x & \text{otherwise,} \end{cases} \quad (60)$$

where

$$A_- = \frac{\eta(b - \alpha_-)}{\alpha_+(\alpha_+ - \alpha_-)} \bar{x}^{b-\alpha_+} - \frac{1 - \alpha_-}{\alpha_+(\alpha_+ - \alpha_-)\beta_1} \bar{x}^{1-\alpha_+}, \quad (61)$$

$$A_+ = \frac{\eta(\alpha_+ - b)}{\alpha_-(\alpha_+ - \alpha_-)} \bar{x}^{b-\alpha_-} - \frac{\alpha_+ - 1}{\alpha_-(\alpha_+ - \alpha_-)\beta_1} \bar{x}^{1-\alpha_-}, \quad (62)$$

the  $x$  value at which the financial wealth is zero is

$$\bar{x} = \left( \frac{\left( \frac{\eta - \hat{\eta}}{b} \zeta^{b - \alpha_-} - \frac{\eta}{\alpha_-} \right) (\alpha_+ - b) \beta_1}{\left( \zeta^{1 - \alpha_-} - \frac{1}{\alpha_-} \right) (\alpha_+ - 1)} \right)^\gamma, \quad (63)$$

the optimal retirement boundary

$$\underline{x} = \zeta \bar{x}, \quad (64)$$

where  $\zeta \in (0, 1)$  is the unique solution to  $q(\zeta) = 0$ , where

$$\begin{aligned} q(\zeta) &\equiv \left( \frac{1 - K^{-b}}{b(1 + \delta k^{-b})} \zeta^{b - \alpha_-} - \frac{1}{\alpha_-} \right) \left( \zeta^{1 - \alpha_+} - \frac{1}{\alpha_+} \right) (\alpha_+ - b) (\alpha_- - 1) \\ &\quad - \left( \frac{1 - K^{-b}}{b(1 + \delta k^{-b})} \zeta^{b - \alpha_+} - \frac{1}{\alpha_+} \right) \left( \zeta^{1 - \alpha_-} - \frac{1}{\alpha_-} \right) (\alpha_- - b) (\alpha_+ - 1), \end{aligned} \quad (65)$$

and

$$\alpha_+ = \frac{\beta_1 - \beta_2 + \frac{1}{2}\beta_3 + \sqrt{(\beta_1 - \beta_2 + \frac{1}{2}\beta_3)^2 + 2\beta_2\beta_3}}{\beta_3}. \quad (66)$$

Then given the new dual variable  $x_t$  and the new dual value function, the rest of the form of the solution are given by (48) through (56) in Theorem 2.

The following lemmas are useful for proving Theorems 2 and 3.

**Lemma 3** (*Dominated Convergence Theorem*) Suppose that a.s.-convergent sequences of random variables  $X_n \rightarrow X$  and  $Y_n \rightarrow Y$  satisfy  $0 \leq X_n \leq Y_n$  and  $E[Y_n] \rightarrow E[Y] < \infty$ . Then  $E[X_n] \rightarrow E[X]$ .

PROOF OF LEMMA 3: Since  $0 \leq X_n \leq Y_n$ , by Fatou's lemma  $\liminf E[X_n] \geq E[X]$  and  $\liminf E[Y_n - X_n] \geq E[Y - X]$ . These inequalities imply that both  $\limsup E[X_n] \geq E[X]$  and  $\liminf E[X_n] \leq E[X]$  since  $E[Y_n] \rightarrow E[Y] < \infty$ . Therefore, we must have  $E[X_n] \rightarrow E[X]$ .  $\square$

Next, let

$$\psi(x) \equiv A_+ x^{\alpha_-} - \eta \frac{x^b}{b} + \frac{1}{\beta_1} x \quad (67)$$

and

$$\hat{\psi}(x) \equiv -\hat{\eta} \frac{x^b}{b}, \quad (68)$$

where  $A_+$  is as defined in Theorem 2.

**Lemma 4** *In Theorem 2, suppose  $\nu > 0$ ,  $\beta_1 > 0$ , and  $\beta_2 > 0$ . Then*

- (i).  $\hat{\psi}(x)$  is strictly decreasing and strictly convex;
- (ii).  $\psi(x)$  is strictly convex and  $\psi_x(x) \leq \frac{1}{\beta_1}$ ;
- (iii).  $\forall x \geq 0$ , we have  $\psi(x) \geq \hat{\psi}(x)$  and  $\forall x \geq \underline{x}$ , we have  $\psi_x(x) \geq \hat{\psi}_x(x)$ .
- (iv). Given (41), there exists a unique solution  $x_0 > 0$  to (43). In addition,  $W_t^*$  defined in (53) satisfies the borrowing constraint (9).

PROOF OF LEMMA 4: (i). This follows from direct differentiation. (ii). These results also follow from direct differentiation, noting that  $A_+ > 0$  and  $\alpha_- < 0$ . (iii). This follows from a similar argument to that for Part (ii) of Lemma 5 below. (iv). By (44), (67), and (68), we have

$$\varphi(x, R) = \begin{cases} \hat{\psi}(x) & \text{if } R = 1 \text{ or } x \leq \underline{x} \\ \psi(x) & \text{otherwise.} \end{cases} \quad (69)$$

By Part (i), Part (ii), and  $\psi_x(\underline{x}) = \hat{\psi}_x(\underline{x})$ ,  $\varphi'(x)$  is continuous and strictly increasing in  $x$ . By inspection of (67) and (68),  $\varphi_x(x, R)$  takes on all values that are less than or equal to  $\frac{1}{\beta_1}$ . Since  $y_0 > 0$ , there exists a unique solution  $x_0 > 0$  to (43) for each  $W_0 \geq -\frac{1}{\beta_1}$ . Also, since  $\varphi_x(x, R) \leq \frac{1}{\beta_1}$ , (53) implies that  $W_t^* > -(1 - R_t)\frac{y_t}{\beta_1}, \forall t \geq 0$ .  $\square$

Next, let

$$\psi(x) \equiv A_+x^{\alpha_-} + A_-x^{\alpha_+} - \eta\frac{x^b}{b} + \frac{1}{\beta_1}x \quad (70)$$

and

$$\hat{\psi}(x) \equiv -\hat{\eta}\frac{x^b}{b}, \quad (71)$$

where  $A_+$  and  $A_-$  are as defined in Theorem 3.

**Lemma 5** *In Problem 3, suppose  $\nu > 0$ ,  $\beta_1 > 0$ , and  $\beta_2 > 0$ . Suppose there exists a solution  $\zeta \in (0, 1)$  to equation (65). Then*

- (i).  $\hat{\psi}(x)$  is strictly convex and strictly decreasing for  $x \geq 0$ ;

(ii).  $\forall x \leq \bar{x}$  we have  $\psi(x) \geq \hat{\psi}(x)$ ,  $\forall x \in [\underline{x}, \bar{x}]$  we have  $\psi_x(x) \geq \hat{\psi}_x(x)$  and

$$\underline{x} < \left( \frac{1 - K^{-b}}{b} \right)^\gamma. \quad (72)$$

(iii).

$$A_- < 0, \quad A_+ > 0, \quad \bar{x} > \left( \frac{(1 - \alpha_-)(1 + \delta k^{-b})}{b - \alpha_-} \right)^\gamma.$$

(iv).  $\psi(x)$  is strictly convex and strictly decreasing for  $x < \bar{x}$ .

(v). Given  $W_0 > 0$ , there exists a unique solution  $x_0 > 0$  to (59). In addition,  $W_t^*$  defined in (53) satisfies the borrowing constraint (10).

PROOF OF LEMMA 5: (i).  $\gamma > 0$  implies that  $b = 1 - 1/\gamma < 1$ . Then since  $\nu > 0$ , direct differentiation shows that  $\hat{\psi}(x)$  is strictly convex and strictly decreasing for  $x > 0$ .

(ii). First, since  $\nu > 0$ ,  $\beta_1 > 0$ , and  $\beta_2 > 0$ , it is straightforward to use the definitions of  $\alpha_+$  and  $\alpha_-$  show that

$$\alpha_+ > 1 > b > \alpha_-, \quad \alpha_- < 0. \quad (73)$$

Recall the definitions of  $\psi(x)$  and  $\hat{\psi}(x)$  in (70) and (71). Let

$$h(x) \equiv \psi(x) - \hat{\psi}(x).$$

It can be easily verified that

$$\frac{1}{2}\beta_3 x^2 \hat{\psi}_{xx}(x) - (\beta_1 - \beta_2)x \hat{\psi}_x(x) - \beta_2 \hat{\psi}(x) - (K^{-b} + \delta k^{-b}) \frac{x^b}{b} = 0, \quad (74)$$

and

$$\frac{1}{2}\beta_3 x^2 \psi_{xx}(x) - (\beta_1 - \beta_2)x \psi_x(x) - \beta_2 \psi(x) - (1 + \delta k^{-b}) \frac{x^b}{b} + x = 0, \quad (75)$$

with

$$\psi(\underline{x}) = \hat{\psi}(\underline{x}), \quad (76)$$

$$\psi_x(\underline{x}) = \hat{\psi}_x(\underline{x}), \quad (77)$$

$$\psi_x(\bar{x}) = 0 \quad (78)$$

and

$$\psi_{xx}(\bar{x}) = 0. \quad (79)$$

Then by (74) and (75),  $h(x)$  must satisfy

$$\frac{1}{2}\beta_3x^2h'' - (\beta_1 - \beta_2)xh' - \beta_2h = \frac{1 - K^{-b}}{b}x^b - x. \quad (80)$$

By (76)-(78) and the fact that  $\hat{\psi}(x)$  is monotonically decreasing for  $x > 0$ , we have

$$h(\underline{x}) = 0, \quad h'(\underline{x}) = 0, \quad h'(\bar{x}) > 0. \quad (81)$$

Differentiating (80) once, we obtain

$$\frac{1}{2}\beta_3x^2h''' + (\beta_3 - \beta_1 + \beta_2)xh'' - \beta_1h' = (1 - K^{-b})x^{b-1} - 1. \quad (82)$$

We consider two possible cases.

Case 1:  $(1 - K^{-b})\underline{x}^{b-1} - 1 < 0$ . In this case, the RHS of equation (82) is negative. Since  $\beta_1 > 0$ ,  $h'(x)$  cannot have any interior nonpositive minimum. To see this, suppose  $\hat{x} \in (\underline{x}, \bar{x})$  achieves an interior minimum with  $h'(\hat{x}) \leq 0$ . Then we would have  $h'''(\hat{x}) > 0$  and  $h''(\hat{x}) = 0$ , which implies that the LHS is positive: a contradiction. Since  $h'(\underline{x}) = 0$ ,  $h'(\bar{x}) > 0$ , we must have  $h'(x) > 0$  for any  $x \in (\underline{x}, \bar{x}]$  because otherwise there would be an interior nonnegative minimum. Then the fact that  $h(\underline{x}) = 0$  implies that  $h(x) > 0$  for any  $x \in (\underline{x}, \bar{x}]$ . Since  $h'(x) > 0$  for any  $x \in (\underline{x}, \bar{x}]$  and  $h'(\underline{x}) = 0$ , we must have  $h''(\underline{x}) \geq 0$ . In addition, if  $h''(\underline{x})$  were equal to 0, then we would have  $h'''(\underline{x}) < 0$  by (81) and (82) since  $(1 - K^{-b})\underline{x}^{b-1} - 1 < 0$ . However, this would contradict the fact that  $h'(x) > 0$  for any  $x \in (\underline{x}, \bar{x}]$  and  $h'(\underline{x}) = 0$ . Therefore, we must have  $h''(\underline{x}) > 0$ . Then (80), (81) and  $h''(\underline{x}) > 0$  imply that

$$\underline{x} < \left( \frac{1 - K^{-b}}{b} \right)^\gamma,$$

Case 2:  $(1 - K^{-b})\underline{x}^{b-1} - 1 \geq 0$ . In this case, we must have  $0 < b < 1$  because  $K > 1$ . Therefore  $\underline{x} \leq (1 - K^{-b})^\gamma < \left( \frac{1 - K^{-b}}{b} \right)^\gamma$ . This implies that  $h''(\underline{x}) > 0$  by (80) and (81). Therefore there exists  $\epsilon > 0$  such that  $h'(x) > 0$  for any  $x \in (\underline{x}, \underline{x} + \epsilon]$  because  $h'(\underline{x}) = 0$ . The RHS of equation (82)

is monotonically decreasing in  $x$ . Let  $x^*$  be such that the RHS of (82) is 0. Then for any  $x \leq x^*$ , the RHS is nonnegative and thus  $h'(x)$  cannot have any interior nonnegative (local) maximum in  $[\underline{x}, x^*]$  for similar reasons to those in Case 1. Thus there cannot exist any  $\hat{x} \in (\underline{x} + \epsilon, x^*]$  such that  $h'(\hat{x}) \leq 0$ . If  $x^* < \bar{x}$ , then for any  $x \in (x^*, \bar{x}]$ , the RHS is non-positive and thus  $h'(x)$  cannot have any interior non-positive (local) minimum in  $(x^*, \bar{x}]$ . Thus there cannot exist any  $\hat{x} \in (x^*, \bar{x}]$  such that  $h'(\hat{x}) \leq 0$ . Therefore, there cannot exist any  $\hat{x} \in (\underline{x}, \bar{x})$  such that  $h'(\hat{x}) \leq 0$  and thus we have  $h'(x) > 0$  and  $h(x) > 0$  for any  $x \in (\underline{x}, \bar{x}]$ .

Now we show for both cases,  $h(x) > 0$  for any  $x < \underline{x}$ . (72) implies that the RHS of (80) is positive for  $x < \underline{x}$  and  $h$  cannot achieve an interior positive maximum for  $x < \underline{x}$ . On the other hand,  $h''(\underline{x}) > 0$ ,  $h''(x)$  is continuous at  $\underline{x}$ , and  $h'(\underline{x}) = 0$  imply that there exists an  $\epsilon > 0$  such that

$$\forall x \in [\underline{x} - \epsilon, \underline{x}], \quad h'(x) < 0. \quad (83)$$

Thus  $\forall x \in [\underline{x} - \epsilon, \underline{x})$ ,  $h(x) > 0$ . Therefore  $\forall x < \underline{x}$ ,  $h(x) > 0$ , since otherwise  $h$  would achieve an interior positive maximum in  $(0, \underline{x})$ .

(iii). It can be shown that

$$A_+ = \frac{(\eta - \hat{\eta})(\alpha_+ - b)}{b(\alpha_+ - \alpha_-)} \underline{x}^{b-\alpha_-} - \frac{(\alpha_+ - 1)}{(\alpha_+ - \alpha_-)\beta_1} \underline{x}^{1-\alpha_-}$$

and

$$\eta = \frac{(\alpha_+ - 1)(1 - \alpha_-)(1 + \delta k^{-b})}{(\alpha_+ - b)(b - \alpha_-)\beta_1}. \quad (84)$$

(72) then implies that  $A_+ > 0$ . Since we also have (62),  $\bar{x}$  must satisfy

$$\bar{x} > \left( \frac{\eta(\alpha_+ - b)\beta_1}{\alpha_+ - 1} \right)^\gamma.$$

Since

$$\frac{\alpha_+ - b}{\alpha_+ - 1} > \frac{b - \alpha_-}{1 - \alpha_-},$$

we have

$$\bar{x} > \left( \frac{\eta(b - \alpha_-)\beta_1}{1 - \alpha_-} \right)^\gamma,$$

which (by the definition (61)) implies that  $A_- < 0$ .

(iv). Differentiating (70) twice, we have, for  $x < \bar{x}$ ,

$$\begin{aligned}\psi_{xx}(x) &= (A_- \alpha_+ (\alpha_+ - 1) x^{\alpha_+ - b} + A_+ \alpha_- (\alpha_- - 1) x^{\alpha_- - b} - \eta(b - 1)) x^{b-2} \\ &> \psi_{xx}(\bar{x}) (x/\bar{x})^{b-2} = 0,\end{aligned}\tag{85}$$

where the inequality follows from the fact that

$$\frac{d}{dx} [A_- \alpha_+ (\alpha_+ - 1) x^{\alpha_+ - b} + A_+ \alpha_- (\alpha_- - 1) x^{\alpha_- - b}] < 0,$$

which is implied by  $A_- < 0$ ,  $A_+ > 0$  and (73), and the last equality in (85) follows from  $\psi_{xx}(\bar{x}) = 0$ . Thus  $\psi(x)$  is strictly convex  $\forall x < \bar{x}$ . Since  $\psi_x(\bar{x}) = 0$  and  $\forall x < \bar{x}$ ,  $\psi_{xx}(x) > 0$ , we must also have  $\forall x < \bar{x}$ ,  $\psi_x(x) < 0$ .

(v). By (60), (70), and (71), we have

$$\varphi(x, R) = \begin{cases} \hat{\psi}(x) & \text{if } R = 1 \text{ or } x \leq \underline{x} \\ \psi(x) & \text{otherwise.} \end{cases}\tag{86}$$

By Part (i), Part (iv), and  $\psi_x(\underline{x}) = \hat{\psi}_x(\underline{x})$ ,  $\varphi'(x)$  is continuous and strictly increasing in  $x$ . By inspection of (70) and (71),  $\varphi_x(x, R)$  takes on all non-positive values. Since  $y_0 > 0$ , there exists a unique solution  $x_0 > 0$  to (59) for each  $W_0 > 0$ . Also, since  $\varphi_x(x, R) < 0$ , (53) implies that  $W_t^* > 0, \forall t \geq 0$ .  $\square$

While the dual approach yields almost explicit solutions, it is simpler to show the optimality of the candidate policies in the primal for these combined optimal stopping and optimal control problems.

**Lemma 6** *Given the definitions in Theorem 2 and Theorem 3,*

1.  $M_t$  as defined in (101) is a supermartingale for any feasible policy and a martingale for the claimed optimal policy.
- 2.

$$\lim_{t \rightarrow \infty} E[(1 - R_t) e^{-(\rho + \delta)t} (1 - \gamma) V(W_t, y_t, 0)] \geq 0,\tag{87}$$

with equality for the claimed optimal policy.

PROOF OF LEMMA 6:

(i) Define  $\bar{W} = -y\varphi_x(\underline{x}, 0)$ . Then for any  $W \geq 0$ ,

$$V(W, y, 0) \geq V(W, y, 1), \quad (88)$$

with equality for  $W \geq \bar{W}$ . This can be shown as follows:

Let  $x$  and  $x^R$  be such that  $-y\varphi_x(x, 0) = W$  and  $-y\varphi_x(x^R, 1) = W$ . Then we have

$$\varphi(x, 0) - \varphi(x^R, 1) \geq \varphi(x, 1) - \varphi(x^R, 1) \geq \varphi_x(x^R, 1)(x - x^R) = x\varphi_x(x, 0) - x^R\varphi_x(x^R, 1),$$

where the first inequality follows from  $\varphi(x, 0) \geq \varphi(x, 1)$  by Lemmas 4 and 5 and the second inequality from the convexity of  $\varphi(x, 1)$ . After rearranging, we obtain (88).

Applying the generalized Itô's lemma to  $M_t$  defined in the proofs of Theorems 2 and 3, we have

$$\begin{aligned} M_t &= M_0 \\ &+ \int_0^t (1 - R_s) \left\{ e^{-(\rho+\delta)s} \left( \frac{c_s^{1-\gamma}}{1-\gamma} + \delta \frac{(kB_s)^{1-\gamma}}{1-\gamma} \right) + E \left[ d \left( e^{-(\rho+\delta)s} V(W_s, y_s, 0) \right) \right] \right\} ds \\ &+ \int_0^t e^{-(\rho+\delta)s} (V(W_s, y_s, 1) - V(W_s, y_s, 0)) dR_s \\ &+ \int_0^t (1 - R_s) e^{-(\rho+\delta)s} (V_W(W_s, y_s, 0) \theta_s^\top \sigma + y_s V_y(W_s, y_s, 0) \sigma_y^\top) dZ_s, \end{aligned} \quad (89)$$

By the definitions of  $V$ ,  $\varphi$ ,  $(c^*, B^*, R^*, W^*)$ , and the fact that  $\varphi(x, 0)$  satisfies (74)-(77), we obtain that the first integral is always non-positive for any feasible policy  $(c, B, \theta, R)$  and is equal to zero for the claimed optimal policy  $(c^*, B^*, \theta^*, R^*)$ . By (88), the third term in (89) is always non-positive for every feasible retirement policy  $R_t$  and equal to zero for the claimed optimal policy  $R_t^*$ . In addition, using the expressions for the claimed optimal  $\theta_t^*$ ,  $V$ ,  $B_t^*$ , and  $W_t^*$ , we have that under the claimed optimal policy, the stochastic integral is a martingale because (1)  $y_t$  is a geometric Brownian motion; (2) in Theorem 2,  $x_t$  is also a geometric Brownian motion; (3) in Theorem 3,  $x_t$  is bounded between  $\underline{x}$  and  $\bar{x}$  before retirement, and  $x_t$  is also a geometric Brownian motion after retirement. This shows that  $M_t$  is a local supermartingale for all feasible policies and a martingale for the claimed optimal policy.

If  $\gamma < 1$ , then  $V(W_t, y_t, 0) > 0$  and thus

$$\lim_{t \rightarrow \infty} E[(1 - R_t)e^{-(\rho+\delta)t}(1 - \gamma)V(W_t, y_t, 0)] \geq 0.$$

In addition,  $M_t$  is always nonnegative and thus a supermartingale.

If  $\gamma > 1$ , we divide the proof that  $M_t$  is actually a supermartingale for any feasible policy into two parts: One for Theorem 2 and the other for Theorem 3.

(A) For Theorem 2, consider an investor who has an initial endowment of  $(W_0, (1 + \epsilon)y_0)$  ( $\epsilon > 0$ ) but follows the same strategy  $(c, B, \theta, R)$  for an investor who has an initial endowment of  $(W_0, y_0)$  and saves the additional income until retirement, and follows the optimal strategy given the implied wealth afterwards. Let the implied wealth process be  $W_t^\epsilon$ , which converges to  $W_t$  as  $\epsilon \rightarrow 0$ . By (89), there exists a series of increasing stopping times  $\tau_n \rightarrow \infty$  such that

$$\begin{aligned} & V(W_0, (1 + \epsilon)y_0, 0) \\ \geq & E \int_0^{\tau_n \wedge t} e^{-(\rho+\delta)s} \left[ (1 - R_s) \left( \frac{c_s^{1-\gamma}}{1-\gamma} + \delta \frac{(kB_s)^{1-\gamma}}{1-\gamma} \right) ds + V(W_s^\epsilon, (1 + \epsilon)y_s, 1) dR_s \right] \\ & + E[(1 - R_{\tau_n \wedge t})e^{-(\rho+\delta)(\tau_n \wedge t)} V(W_{\tau_n \wedge t}^\epsilon, (1 + \epsilon)y_{\tau_n \wedge t}, 0)]. \end{aligned} \quad (90)$$

Since the integrand in the integral of (90) is always negative, this integral monotonically decreases in time. In addition,

$$\begin{aligned} 0 & \geq (1 - R_t)e^{-(\rho+\delta)t} V(W_t^\epsilon, (1 + \epsilon)y_t, 0) \\ & \geq e^{-(\rho+\delta)t} V\left(-\frac{y_t}{\beta_1}, (1 + \epsilon)y_t, 0\right) \\ & = V\left(-\frac{1}{\beta_1}, (1 + \epsilon), 0\right) e^{-(\rho+\delta)t} y_t^{1-\gamma} \end{aligned} \quad (91)$$

$$\geq V\left(-\frac{1}{\beta_1}, (1 + \epsilon), 0\right) N_t, \quad (92)$$

where

$$N_t \equiv e^{-\frac{1}{2}(1-\gamma)^2\sigma_y^2 t + (1-\gamma)\sigma_y Z_t} \quad (93)$$

is a martingale with  $E[N_t] = 1$ , the second inequality follows from  $V$  negative and increasing in  $W$  and  $W_t^\epsilon > W_t > -\frac{y_t}{\beta_1}$ , the equality follows from the form of  $V$  as defined by (55) and (56), and the

last inequality follows from  $V(-\frac{1}{\beta_1}, (1 + \epsilon), 0) < 0$  and  $\beta_2 > 0$ . In addition,  $V(-\frac{1}{\beta_1}, (1 + \epsilon), 0) > -\infty$ .

Therefore, taking  $n \rightarrow \infty$  in (90), by the monotone convergence theorem for the first term and Lemma 3 (a generalized dominated convergence theorem) for the second term, we have

$$\begin{aligned} & V(W_0, (1 + \epsilon)y_0, 0) \\ \geq & E \int_0^t e^{-(\rho+\delta)s} \left[ (1 - R_s) \left( \frac{c_s^{1-\gamma}}{1-\gamma} + \delta \frac{(kB_s)^{1-\gamma}}{1-\gamma} \right) ds + V(W_s^\epsilon, (1 + \epsilon)y_s, 1) dR_s \right] \\ & + E[(1 - R_t)e^{-(\rho+\delta)t}V(W_t^\epsilon, (1 + \epsilon)y_t, 0)]. \end{aligned} \quad (94)$$

Next, taking  $\epsilon \rightarrow 0$  in (94), we obtain  $M_0 \geq E[M_t]$  for any  $t \geq 0$ . Since the above argument applies to any time  $s \leq t$ , we have  $M_s \geq E[M_t]$  for any  $t \geq s$  and thus  $M_t$  is a supermartingale for every feasible policy.

(B) For Theorem 3, by (89), there exists an increasing sequence of stopping times  $\tau_n \rightarrow \infty$  such that  $M_0 \geq E[M_{\tau_n \wedge t}]$ , i.e.,

$$\begin{aligned} & V(W_0, y_0, 0) \\ \geq & E \int_s^{\tau_n \wedge t} e^{-(\rho+\delta)s} \left[ (1 - R_s) \left( \frac{c_s^{1-\gamma}}{1-\gamma} + \delta \frac{(kB_s)^{1-\gamma}}{1-\gamma} \right) ds + V(W_s, y_s, 1) dR_s \right] \\ & + E[(1 - R_{\tau_n \wedge t})e^{-(\rho+\delta)(\tau_n \wedge t)}V(W_{\tau_n \wedge t}, y_{\tau_n \wedge t}, 0)]. \end{aligned} \quad (95)$$

Since the integrand in the integral of (95) is always negative, this integral is monotonically decreasing in time. In addition,

$$\begin{aligned} 0 & \geq (1 - R_t)e^{-(\rho+\delta)t}V(W_t, y_t, 0) \\ & \geq e^{-(\rho+\delta)t}V(0, y_t, 0) \\ & = V(0, 1, 0)e^{-(\rho+\delta)t}y_t^{1-\gamma} \end{aligned} \quad (96)$$

$$\geq V(0, 1, 0)N_t, \quad (97)$$

where  $N_t$ , as defined in (93), is a martingale with  $E[N_t] = 1$ , the second inequality follows from  $V$  negative and increasing in  $W$  and  $W_t > 0$ , the equality follows from the form of  $V$  as defined by (55) and (56), and the last inequality follows from  $V(0, 1, 0) < 0$  and  $\beta_2 > 0$ . In addition,

$V(0, 1, 0) > -\infty$ .

Therefore, taking  $n \rightarrow \infty$  in (95), by the monotone convergence theorem for the first term and Lemma 3 (a generalized dominated convergence theorem) for the second term, we have

$$\begin{aligned} & V(W_0, y_0, 0) \\ \geq & E \int_0^t e^{-(\rho+\delta)s} \left[ (1 - R_s) \left( \frac{c_s^{1-\gamma}}{1-\gamma} + \delta \frac{(kB_s)^{1-\gamma}}{1-\gamma} \right) ds + V(W_s, y_s, 1) dR_s \right] \\ & + E[(1 - R_t)e^{-(\rho+\delta)t}V(W_t, y_t, 0)]. \end{aligned} \quad (98)$$

That is:  $M_0 \geq E[M_t]$  for any  $t \geq 0$ . Since the above argument applies to any time  $s \leq t$ , we have  $M_s \geq E[M_t]$  for any  $t \geq s$  and thus  $M_t$  is a supermartingale for all feasible policies.

(ii) First, we show  $(1 - \gamma)V(W, y, 0) \geq 0$  for every feasible policy.  $V_W(W, y, 0) = y^{-\gamma}x > 0$  and thus  $V(W, y, 0)$  increases in  $W$ . If  $\gamma < 1$ , then  $V(W, y, 0) \geq 0$  because  $V(W, y, 0) \geq V(W, y, 1) \geq 0$ . If  $\gamma > 1$ , then  $V(W, y, 0) < 0$  because  $V(W, y, 0) \leq V(\bar{W}, y, 0) = V(\bar{W}, y, 1) < 0$ . Therefore,  $(1 - \gamma)V(W, y, 0) \geq 0$  for every feasible policy. It follows that

$$\begin{aligned} 0 & \leq \lim_{t \rightarrow \infty} E[(1 - R_t)e^{-(\rho+\delta)t}(1 - \gamma)V(W_t, y_t, 0)] \\ & = \lim_{t \rightarrow \infty} E[(1 - R_t)e^{-(\rho+\delta)t}y_t^{1-\gamma}(1 - \gamma)(\varphi(x_t, 0) - x_t\varphi_x(x_t, 0))] \\ & \leq \lim_{t \rightarrow \infty} E[L_1e^{-(\rho+\delta)t}y_t^{1-\gamma} + \hat{\eta}e^{-(\rho+\delta)t/\gamma}\xi_t^b] \\ & = 0, \end{aligned} \quad (99)$$

where the second inequality follows from (1) the definition of  $\varphi$  in Theorem 2,  $A_+ > 0$ ,  $\alpha_- < 0$ , and  $x_t > \underline{x}$ ; and (2) the fact that in Theorem 3,  $x_t$ ,  $\varphi(x_t)$ , and  $\varphi_x(x_t)$  are all bounded, while  $R_t = 1$  for  $t > \tau^*$ . The last equality in (99) follows from the conditions that  $\nu > 0$  and  $\beta_2 > 0$ .

Therefore, for the claimed optimal policy, we obtain

$$\lim_{t \rightarrow \infty} E[(1 - R_t)e^{-(\rho+\delta)t}V(W_t, y_t, 0)] = 0.$$

For any feasible policy, if  $\gamma < 1$ , then  $V(W, y, 0) > 0$  and therefore (87) holds. If  $\gamma > 1$ , since  $\beta_2 > 0$ , we have  $\lim_{t \rightarrow \infty} E[e^{-(\rho+\delta)t} y_t^{1-\gamma}] = 0$ . Therefore, taking the limit as  $t \rightarrow \infty$  in (91),

$$\lim_{t \rightarrow \infty} E[(1 - R_t)e^{-(\rho+\delta)t} V(W_t, (1 + \epsilon)y_t, 0)] = 0,$$

which implies that (87) holds, after taking the limit as  $\epsilon \rightarrow 0$ . Similarly taking the limit as  $t \rightarrow \infty$  in (96), we have that (87) also holds. This completes the proof.  $\square$

**Lemma 7** *Suppose  $\nu > 0$ ,  $\beta_1 > 0$ , and  $\beta_2 > 0$ . Then there exists a unique solution  $\zeta^* \in (0, 1)$  to equation (65) and*

$$\zeta^* < \bar{\zeta} = \text{Min}\left(\left(\frac{1 - K^{-b}}{b(1 + \delta k^{-b})}\right)^\gamma, 1\right).$$

PROOF OF LEMMA 7: Since  $\nu > 0$ ,  $\beta_1 > 0$ , and  $\beta_2 > 0$ ,

$$\alpha_+ > 1 > b > \alpha_-, \quad \alpha_- < 0.$$

Next, since  $\zeta^{b-\alpha_+}$  dominates  $\zeta^{1-\alpha_+}$  as  $\zeta \rightarrow 0$ , we have

$$\lim_{\zeta \rightarrow 0} q(\zeta) = \lim_{\zeta \rightarrow 0} -\frac{1 - K^{-b}}{b(1 + \delta k^{-b})}(\alpha_- - b)(\alpha_+ - 1)\zeta^{b-\alpha_+} = +\infty.$$

Next, it is easy to verify that

$$q(1) = -\frac{(\alpha_+ - 1)(\alpha_- - 1)(\alpha_+ - \alpha_-)(K^{-b} + \delta k^{-b})}{\alpha_+ \alpha_- (1 + \delta k^{-b})} < 0.$$

Now suppose  $\hat{\zeta} = \left(\frac{1 - K^{-b}}{b(1 + \delta k^{-b})}\right)^\gamma < 1$ . Then we have  $\frac{1 - K^{-b}}{b(1 + \delta k^{-b})}\hat{\zeta}^{b-\alpha_+} - \frac{1}{\alpha_+} = \hat{\zeta}^{1-\alpha_+} - \frac{1}{\alpha_+}$ ,  $\frac{1 - K^{-b}}{b(1 + \delta k^{-b})}\hat{\zeta}^{b-\alpha_-} - \frac{1}{\alpha_-} = \hat{\zeta}^{1-\alpha_-} - \frac{1}{\alpha_-}$ , and  $\hat{\zeta}^{1-\alpha_+} > 1 > \frac{1}{\alpha_+}$ . It follows that

$$q(\hat{\zeta}) = -\frac{1}{\gamma}\left(\hat{\zeta}^{1-\alpha_+} - \frac{1}{\alpha_+}\right)\left(\hat{\zeta}^{1-\alpha_-} - \frac{1}{\alpha_-}\right)(\alpha_+ - \alpha_-) < 0.$$

Then by continuity of  $q$ , there exists a solution  $\zeta^* \in (0, \bar{\zeta})$  such that  $q(\zeta^*) = 0$ . Suppose there exists another solution  $\hat{\zeta} \in [0, 1)$  such that  $q(\hat{\zeta}) = 0$ . Let  $V(W, y, 0)$  and  $\bar{W}$  be the value function and boundary respectively corresponding to  $\zeta^*$  and  $\hat{v}(W, y, 0)$  and  $\hat{\bar{W}}$  be the value function and

boundary respectively corresponding to  $\hat{\zeta}$ . Without loss of generality, suppose  $\bar{W} > \hat{W}$ . Since  $\hat{W}$  is the retirement boundary, the value function corresponding to  $\hat{\zeta}$  for  $\bar{W} > W > \hat{W}$  is equal to  $V(W, y, 1)$ . However, Lemma 5 implies that  $V(W, y, 0) > V(W, y, 1)$  for any  $W < \bar{W}$ . This implies that  $\hat{W}$  cannot be the optimal retirement boundary, which contradicts Theorem 3. Therefore, the solution to equation (65) is unique.  $\square$

We are now ready to prove Theorems 2 and 3.

PROOF OF THEOREMS 2 AND 3: If  $R_0 = 1$ , then Problems 2 and 3 are identical to Problem 1. Therefore, the optimality of the claimed optimal strategy follows from Theorem 1. From now on, we assume w.l.o.g. that  $R_0 = 0$ . It is tedious but straightforward to use the generalized Itô's lemma, equations (44)-(53), and (60)-(66) to verify that the claimed optimal strategy  $W_t^*$ ,  $c_t^*$ ,  $\theta_t^*$ , and  $R_t^*$  in these two theorems satisfy the budget constraint (4). In addition, by Lemmas 4 and 5,  $x_0$  exists and is unique and  $W_t^*$  satisfies the borrowing constraint in each problem. Furthermore, by Lemma 7, there is a unique solution to (65).

By Doob's optional sampling theorem, we can restrict attention w.l.o.g. to the set of feasible policies that implement the optimal policy stated in Theorem 1 after retirement, and after integrating out the mortality risk, the utility function for such a strategy can be written as

$$E \int_0^\infty e^{-(\rho+\delta)s} \left[ (1 - R_s) \left( \frac{c_s^{1-\gamma}}{1-\gamma} + \delta \frac{(kB_s)^{1-\gamma}}{1-\gamma} \right) ds + V(W_s, y_s, 1) dR_s \right]. \quad (100)$$

Accordingly, define

$$M_t = \int_0^t e^{-(\rho+\delta)s} \left[ (1 - R_s) \left( \frac{c_s^{1-\gamma}}{1-\gamma} + \delta \frac{(kB_s)^{1-\gamma}}{1-\gamma} \right) ds + V(W_s, y_s, 1) dR_s \right] + (1 - R_t) e^{-(\rho+\delta)t} V(W_t, y_t, 0). \quad (101)$$

By Lemma 6 in the Appendix,  $M_t$  is a supermartingale for any feasible policy  $(c, B, R, W)$  and a martingale for the claimed optimal policy  $(c^*, B^*, R^*, W^*)$ , which implies that  $M_0 \geq E[M_t]$ , i.e.,

$$V(W_0, y_0, 0) \geq E \int_0^t e^{-(\rho+\delta)s} \left[ (1 - R_s) \left( \frac{c_s^{1-\gamma}}{1-\gamma} + \delta \frac{(kB_s)^{1-\gamma}}{1-\gamma} \right) ds + V(W_s, y_s, 1) dR_s \right] + E[(1 - R_t) e^{-(\rho+\delta)t} V(W_t, y_t, 0)], \quad (102)$$

and with equality for the claimed optimal policy. In addition, by Lemma 6, we also have that

$$\lim_{t \rightarrow \infty} E[(1 - R_t)e^{-(\rho+\delta)t}V(W_t, y_t, 0)] \geq 0,$$

with equality for the claimed optimal policy.

Therefore, taking the limit as  $t \uparrow \infty$  in (102), we have

$$V(W_0, y_0, 0) \geq E \int_0^\infty e^{-(\rho+\delta)s} \left[ (1 - R_s) \left( \frac{c_s^{1-\gamma}}{1-\gamma} + \delta \frac{(kB_s)^{1-\gamma}}{1-\gamma} \right) ds + V(W_s, y_s, 1) dR_s \right],$$

with equality for the claimed optimal policy  $(c^*, B^*, R^*)$ . This completes the proof.  $\square$

Theorems 2 and 3 provide essentially complete solutions, since the solution for  $x_0$  given  $W_0$  requires only a one-dimensional monotone search to solve equations (43) and (59) and for  $\zeta$  one only needs to solve equation (65).

We next provide results on computing the market value of human capital at any point in time, which is useful for understanding much of the economics in the paper.

**Proposition 1** *Consider the optimal policies stated in Theorems 1-3. After retirement, the market value of the human capital is zero. Before retirement, in Theorem 1 the market value of the human capital is*

$$H(y_t, t) = g(t)y_t,$$

where  $y_t$  and  $g(t)$  are given in (1) and (6); in Theorem 2 the market value of the human capital is

$$H(x_t, y_t) = \frac{y_t}{\beta_1} (-\underline{x}^{1-\alpha_-} x_t^{\alpha_- - 1} + 1),$$

where  $y_t$ ,  $\beta_1$ ,  $x_t$ ,  $\underline{x}$ , and  $\alpha_-$  are given in (1), (7), (21), (46), and (47); and in Theorem 3 the market value of the human capital is

$$H(x_t, y_t) = \frac{y_t}{\beta_1} (Ax_t^{\alpha_- - 1} + Bx_t^{\alpha_+ - 1} + 1),$$

where  $y_t$ ,  $\beta_1$ ,  $\alpha_-$ ,  $x_t$ , and  $\alpha_+$ , are given in (1), (7), (47), (58), and (66), and where

$$A = \frac{(1 - \alpha_+) \underline{x}^{1-\alpha_-} \bar{x}^{\alpha_+ - \alpha_-}}{(\alpha_+ - 1) \bar{x}^{\alpha_+ - \alpha_-} - (\alpha_- - 1) \underline{x}^{\alpha_+ - \alpha_-}}$$

and

$$B = \frac{(\alpha_- - 1)\underline{x}^{1-\alpha_-}}{(\alpha_+ - 1)\bar{x}^{\alpha_+ - \alpha_-} - (\alpha_- - 1)\underline{x}^{\alpha_+ - \alpha_-}}.$$

PROOF: The result on the market value of human capital for Problem 1 is directly implied by Lemma 1, Part (i).

For the cases with voluntary retirement, since there is no more labor income after retirement, the market value of human capital after retirement is zero. We next prove the claims for after retirement. Using the expressions of  $H$  and the dynamics of  $x_t$  and  $y_t$ , it can be verified that for  $x_t > \underline{x}$  in Theorem 2 and for  $\underline{x} < x_t < \bar{x}$  in Theorem 3, we have that the change in the market value of human capital plus the flow of labor income will be given by

$$\begin{aligned} & d(\xi_t H(x_t, y_t)) + \xi_t y_t dt \\ = & \xi_t \left( \frac{1}{2} \beta_3 x_t^2 H_{xx} - (\beta_1 - \beta_2 - \beta_3) x_t H_x - \beta_1 H + y_t \right) dt \\ & + \xi_t (x_t H_x \sigma_x^\top + H(\sigma_y^\top - \kappa^\top)) dZ_t. \end{aligned} \quad (103)$$

The drift term in (103) is equal to zero after plugging in the expressions for  $H$  (for Theorem 3, the additional local time term at  $\bar{x}$  from applying the generalized Itô's lemma is also equal to zero because it can be verified that  $H_x(\bar{x}, y_t) = 0$ ). This implies that

$$\mathcal{M}_t \equiv \xi_t H(x_t, y_t) + \int_0^t \xi_s y_s ds$$

is a local martingale. In addition, there exists a constant  $0 < L < \infty$  such that

$$|\xi_t (x_t H_x \sigma_x^\top + H(\sigma_y^\top - \kappa^\top))| < L \xi_t y_t$$

in Theorem 2 since  $\alpha_- < 0$  and  $x_t > \underline{x}$ , and in Theorem 3 since  $\underline{x} < x_t \leq \bar{x}$ . Since both  $\xi_t$  and  $y_t$  are geometric Brownian motions, we have that  $\mathcal{M}_t$  is actually a martingale. Recall the definition (54) of the optimal retirement time  $\tau^*$ . We have,  $\forall t \leq \tau^*$

$$\xi_t H(x_t, y_t) + \int_0^t \xi_s y_s ds = E_t[\xi_{\tau^*} H(\underline{x}, y_{\tau^*}) + \int_0^{\tau^*} \xi_s y_s ds],$$

which implies that

$$H(x_t, y_t) = \xi_t^{-1} E_t \left[ \int_t^{\tau^*} \xi_s y_s ds \right],$$

since it can be easily verified that  $H(\underline{x}, y) = 0$ . Therefore  $H$  as specified in the proposition is indeed the market value of the future labor income.  $\square$

The following result shows that because of the retirement flexibility human capital may have a negative beta, even when the labor income correlates positively with the market risk.

**Proposition 2** *As the investor's financial wealth  $W$  increases, the investor's human capital  $H$  decreases in both Problem 2 and Problem 3. Furthermore, if  $\sigma_y < \kappa/\gamma$ , then human capital has a negative beta measured relative to any locally mean-variance efficient risky portfolio.*

PROOF: First, as shown in Lemmas 4 and 5, the dual value function  $\varphi$  defined in Theorems 2 and 3 are convex and thus the wealth level  $W_t$  defined in these theorems decreases with the dual variable  $x_t$ . By Proposition 1, given the optimal policy in Theorem 2, differentiating the expression for the human capital with respect to  $x_t$  yields that the human capital is increasing in  $x_t$ . Therefore, human capital decreases with the financial wealth  $W$  for Problem 2. For Problem 3, differentiating the human capital  $H$  with respect to  $x$ , we have that before retirement

$$\begin{aligned} \frac{\partial H(x, y)}{\partial x} &= \frac{y}{\beta_1} (A(\alpha_- - 1)x^{\alpha_- - 1} + B(\alpha_+ - 1)x^{\alpha_+ - 1}) \\ &= \frac{y}{\beta_1} \frac{(\alpha_+ - 1)(1 - \alpha_-)\underline{x}^{1 - \alpha_-} x^{\alpha_+ - 2}}{(\alpha_+ - 1)\bar{x}^{\alpha_+ - \alpha_-} - (\alpha_- - 1)\underline{x}^{\alpha_+ - \alpha_-}} \left( \left( \frac{\bar{x}}{x} \right)^{\alpha_+ - \alpha_-} - 1 \right) > 0, \end{aligned} \quad (104)$$

where the second equality follows from the expressions of  $A$  and  $B$  in Proposition 1 and the inequality follows from the fact that  $\alpha_+ > 1 > \alpha_-$  and  $x < \bar{x}$ . Thus, human capital decreases with the financial wealth  $W$  also for Problem 3. Furthermore, since  $\sigma_x = \gamma\sigma_y - \kappa$ , if  $\sigma_y < \kappa/\gamma$ , then as the market risk  $Z_t$  increases,  $x_t$  decreases, and therefore human capital decreases by (104), i.e., human capital has a negative beta.  $\square$

The following result shows that retirement flexibility tends to increase stock investment.

**Proposition 3** *Suppose  $\sigma_y = 0$  and  $\mu > r$ . Then the fraction of total wealth  $W + H$  invested in the risky asset in Problem 2 is greater than that in Problem 1.*

PROOF: By Theorem 1, the fraction of total wealth  $W + H$  invested in the risky asset in Problem 1 is constantly equal to  $\frac{\mu-r}{\gamma\sigma^2}$ .

By Theorem 2 and Lemma 1, we have

$$\frac{\theta}{W + H} = \frac{\mu - r}{\gamma\sigma^2} \frac{\gamma A_+ \alpha_- (\alpha_- - 1) x^{\alpha_- - 1} + \eta x^{b-1}}{-A_+ \alpha_- x^{\alpha_- - 1} + \eta x^{b-1} - \frac{1}{\beta_1} \underline{x}^{1-\alpha_-} x^{\alpha_- - 1}}.$$

Plugging in the expressions for  $A_+$  and  $\underline{x}$  and using the fact that  $\alpha_- < b$ , we have

$$\frac{\gamma A_+ \alpha_- (\alpha_- - 1) x^{\alpha_- - 1} + \eta x^{b-1}}{-A_+ \alpha_- x^{\alpha_- - 1} + \eta x^{b-1} - \frac{1}{\beta_1} \underline{x}^{1-\alpha_-} x^{\alpha_- - 1}} > 1.$$

□

The retirement decision is critical for the investor's consumption and investment policies. The following proposition shows that the presence of borrowing constraint tends to make an investor retire earlier.

**Proposition 4** *The retirement wealth threshold for Problem 2 is higher than that for Problem 3.*

PROOF: Let

$$h(x) = A_+ x^{\alpha_-} + A_- x^{\alpha_+} - \eta \frac{x^b}{b} + \frac{1}{\beta_1} x + \hat{\eta} \frac{x^b}{b},$$

where  $A_+$  and  $A_-$  are defined as in Theorem 3,

$$h_N(x) = A_{+N} x^{\alpha_-} - \eta \frac{x^b}{b} + \frac{1}{\beta_1} x + \hat{\eta} \frac{x^b}{b},$$

where  $A_{+N}$  denotes the coefficient  $A_+$  defined in Theorem 2. Let  $\underline{x}$  be as defined in Theorem 3 and  $\underline{x}_N$  denote the retirement boundary  $\underline{x}$  defined in Theorem 2.

We prove by contradiction. Suppose  $\underline{x} \leq \underline{x}_N$ . By Lemmas 4 and 5, we have  $h(\underline{x}) = h'(\underline{x}) = 0$ ,  $h_N(\underline{x}_N) = h'(\underline{x}_N) = 0$ . From the proof of Lemma 5, and  $h'(x) > 0$  for all  $x \in (\underline{x}, \bar{x}]$  and therefore

$$h(\underline{x}_N) \geq 0 = h_N(\underline{x}_N) \text{ and } h'(\underline{x}_N) \geq 0 = h'_N(\underline{x}_N). \quad (105)$$

The first equation of (105) implies that

$$A_+ \underline{x}_N^{\alpha_-} + A_- \underline{x}_N^{\alpha_+} \geq A_{+N} \underline{x}_N^{\alpha_-},$$

which in turn implies

$$A_+ > A_{+N}, \quad (106)$$

since  $A_- < 0$  as shown in Lemma 5. On the other hand, the second equation of (105) implies that

$$A_+ \alpha_- \underline{x}_N^{\alpha_- - 1} + A_- \alpha_+ \underline{x}_N^{\alpha_+ - 1} \geq A_{+N} \alpha_- \underline{x}_N^{\alpha_- - 1},$$

which in turn implies

$$A_+ < A_{+N}, \quad (107)$$

since  $A_- < 0$  and  $\alpha_+ > 0$ . Result (107) contradicts (106). This shows that we must have  $\underline{x} > \underline{x}_N$ . Since at retirement the financial wealth is equal to  $-y\varphi(\underline{x}_N)$  and  $-y\varphi(\underline{x})$  for Problems 2 and 3 respectively and  $-y\varphi(\underline{x}_N) = \hat{\eta}\underline{x}_N^{-1/\gamma}$ ,  $-y\varphi(\underline{x}) = \hat{\eta}\underline{x}^{-1/\gamma}$ , we must have that the financial wealth level  $W$  at retirement for Problem 2 is higher than that for Problem 3.  $\square$

One measure that is useful for examining the life cycle investment policy is the expected time to retirement. The following proposition shows how to compute this measure.

**Proposition 5** *Suppose that the investor is not retired and  $\frac{1}{2}\sigma_x^2 - \mu_x > 0$ . Then the expected time to retirement for the optimal policy is*

$$E_t[\tau^* | x_t = x] = \frac{\log(x/\underline{x})}{\frac{1}{2}\sigma_x^2 - \mu_x}, \forall x_t > \underline{x}$$

in Theorem 2 and is

$$E_t[\tau^* | x_t = x] = \frac{\underline{x}^m - x^m}{(\frac{1}{2}\sigma_x^2 - \mu_x)m\bar{x}^m} + \frac{\log(x/\underline{x})}{\frac{1}{2}\sigma_x^2 - \mu_x}, \forall x_t \in [\underline{x}, \bar{x}] \quad (108)$$

in Theorem 3, where

$$m = 1 - \frac{2\mu_x}{\sigma_x^2}.$$

PROOF: PROOF OF PROPOSITION 5. First we prove the result for Theorem 2. Recall that

$$dx_t = \mu_x x_t dt + \sigma_x x_t dB_t.$$

Let

$$f(x) \equiv \frac{\log(x/\underline{x})}{\frac{1}{2}\sigma_x^2 - \mu_x}.$$

Then by Itô's lemma, for any stopping time  $\mathcal{T} \geq t$  we have

$$f(x_{\mathcal{T}}) + \int_t^{\mathcal{T}} 1 ds = f(x_t) + \int_t^{\mathcal{T}} \left( \frac{1}{2}\sigma_x^2 x_s^2 f_{xx} + \mu_x x_s f_x + 1 \right) ds + \int_t^{\mathcal{T}} \frac{\sigma_x}{\frac{1}{2}\sigma_x^2 - \mu_x} dZ_s, \quad (109)$$

which implies that  $f(x_{\mathcal{T}}) + \int_t^{\mathcal{T}} 1 ds$  is a martingale, since it can be easily verified that the drift term is zero given the definition of  $f(x)$  and the stochastic integral is a scaled Brownian motion and thus a martingale. Thus, taking  $\mathcal{T} = \tau^*$  and taking expectation in (109), we get

$$f(x) = E_t[\tau^* | x_t = x],$$

since  $x_{\tau^*} = \underline{x}$  and  $f(\underline{x}) = 0$ . A similar argument applies to the case for Theorem 3, noting that when evaluated at  $x = \bar{x}$ , the first derivative of the right hand side of (108) with respect to  $x$  is zero and  $x_t$  is bounded. This completes the proof.  $\square$

The following proposition shows how the expected-time-to-retirement is related to human capital and financial wealth, which can help explain the graphical solutions presented in the previous section.

**Proposition 6** *Suppose that the investor is not retired and  $\frac{1}{2}\sigma_x^2 - \mu_x > 0$ . Then as the expected-time-to-retirement increases, financial wealth decreases and human capital increases.*

PROOF: By Proposition 5, it can be easily verified that the expected-time-to-retirement is increasing in  $x$ . By (104), we have that human capital is increasing in  $x$  and Proposition 2 then implies that financial wealth is decreasing in  $x$ . Therefore, the claims hold.  $\square$

## IV Conclusion

We examine the impact of retirement flexibility and borrowing constraint against future labor income on optimal consumption and investment policy. We solve three alternative models almost explicitly (at least parametrically up to at most a constant) and provide verification theorems that

are proved using a combination of the dual approach and an analysis of the boundary. In addition, we also obtain some interesting comparative statics.

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