OPTIMAL CASUALTY INSURANCE AND REPAIR IN THE PRESENCE OF A SECURITIES MARKET

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Abstract. We build a simple economic model of optimal casualty insurance based on a story about insuring a house. With endogenous repair and a securities market that is complete over states distinguished by security payoffs, we have three main findings in our base model with additively-separable preferences. First, optimal repair depends on security market conditions, with full repair in inexpensive states and little or no repair in expensive states. Second, the optimal insurance payment equals the cost of optimal repair. Third, the agent is not made whole, since the loss is fully compensated only when damage is fully repaired. Weaker versions of the results hold when preferences are not additively-separable. Quite generally, when full repair is optimal it is fully insured.

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Introduction

In the simplest model of optimal insurance, there is actuarially fair insurance, and agents are fully insured in the sense that insurance makes them indifferent about whether a loss occurs or not. As discussed by Arrow (1963) and formalized more fully elsewhere in the literature,† there is less than full insurance in the presence of various frictions such as a cost of processing a claim and informational problems (such as adverse selection and moral
hazard). This paper explores a reason there may optimally be full insurance in some market states and less than full insurance in others. In our model, there is endogenous repair and also a securities market, and the optimal amount of insurance depends on the security market state. This idea is developed in a stylized example with a possible casualty loss on a house. The example has no aggregate actuarial risk and a frictionless securities market for market-level risk. In our benchmark model with additively-separable preferences, there is less than full insurance, but optimal repair is fully insured. Specifically, the optimal insurance policy fully insures the casualty loss in less expensive market states when repair is optimal, but does not pay any compensation in expensive market states in which repair is not optimal (even though there is not any friction or informational asymmetry). This paper can be viewed as an extension of Koehl and Villeneuve (2002), which also has less than full insurance due to endogenous repair, but does not have a stock market and therefore cannot have our main result of contingency of the insurance payment on the state of the market.

We have three main results for additively-separable preferences. First, optimal repair depends on security market conditions, with full repair in inexpensive states and no repair in expensive states. Second, the optimal insurance payment equals the cost of optimal repair; therefore, there is no payment when there is no repair and the optimal repair is fully insured. Third, the agent is not made whole, since the loss is fully compensated only when it is fully repaired.

We compare full insurance with full repair and no insurance with no repair with optimal insurance. Optimal insurance obviously always delivers the highest expected utility or certainty equivalent. If repair is costly enough, no insurance is better than full insurance. As the cost of repair increases, the optimal policy is to repair in fewer and fewer states, so there is less difference between optimal insurance and no insurance. However, the full insurance policy is worse and worse and ultimately infeasible because there is more and more probability the insurance policy will pay for a repair that is not done and eventually the cost of insurance is greater than initial wealth. This just adds noise to financial wealth. As the cost of repair goes to infinity, expected utility for no insurance and no repair and
expected utility for the optimal contract both approach the same limit, while full insurance with full repair becomes infeasible. As the cost of repair decreases, there are smaller and smaller differences between optimal insurance and full insurance. If the cost of repair converges to zero, expected utility for full insurance with full repair and expected utility for the optimal contract both approach the same limit, while the expected utility for no insurance with no repair is smaller.

The main results for additively-separable preferences are also examined for more general preferences that may not be additively separable. It holds generally that the cost of full repair is fully insured. Except in the degenerate case of perfect substitutes, optimal repair in the examples still depends on the state of the securities market, and optimal insurance does not make agents whole except when there is full repair. The case of perfect substitutes is different because housing and other consumption are in effect the same good.

The results of our paper have several empirical predictions. If preferences are additively separable, optimal insurance equals the cost of optimal repair. This predicts that insurance will pay off when there is repair but not when there is no repair. This is consistent with contracts that pay only in kind, for example, with cash going directly to the repair facility and not to the policy-holder. The extent to which there is repair when there is no payment is an interesting empirical question. If insurance pays for full repair and also pays something when there is no repair, it is consistent with non-additively-separable preferences. Probably the most striking prediction of the paper is that payments on insurance contracts depend on financial aggregates as well as the size and nature of the loss. This result does not seem to be consistent with existing insurance contracts and presents an interesting puzzle why this feature of our model is missing in practice.²

The remainder of the paper is organized as follows. Section 1 investigates optimal casualty insurance and repair problems in a stylized model where additively-separable

²Our repair decision is different from the repair or replace decision in auto insurance (in which the insurance could declare the car a total loss), although presumably that decision does depend on market conditions in practice. Index-linked annuities do depend on market conditions as well as the insurable event (longevity), although that setting is different from ours with costly repair.
preferences are assumed. In Section 2, comparisons are made between full insurance with full repair and no insurance with no repair and optimal insurance to examine the efficiency of optimal insurance. Furthermore, we develop an asymptotic result for the benchmark model, when the cost of repair converges to zero or infinity. Section 3 extends the benchmark model to more general preferences. Section 4 concludes the paper. The appendix contains most of the proofs.

1. A stylized model of optimal casualty insurance: benchmark model

There are two points of time, 0 and 1. Each agent has endowment only at time 0: cash and a house of quality-adjusted size \(H\) purchased previously. At time 0, the agent may invest his endowment in securities in a market that is complete over states distinguished by security payoffs, and he may also purchase insurance through a mutual insurance company. Owning the house carries a risk of a casualty loss (this is what the insurance is for). The agent’s casualty loss is given by the random variable \(C\) taking values in \([0, H]\). The objective probability distribution of the loss is common knowledge for all agents, and equals the true population average that will be realized. If \(C = 0\), there is no casualty loss. We assume there is a positive probability of \(C = 0\) conditional on the state of the securities market, given by \(\xi\) defined below. \(C\) may be independent of \(\xi\), but it does not have to be.

Investment in the market and buying insurance are both valued by a nonatomic state-price density\(^3\) \(\xi\) with full support on \(\mathbb{R}_+\). The agent’s budget constraint for terminal financial wealth \(P\) is

\[
W_0 = E[\xi P(\xi, C)].
\]

\(P\) is the total terminal financial payment, which equals the final value of investments plus the insurance payment. Valuing a claim in a complete financial market using a state-price

\(^3\)We take the random state price density \(\xi\) to be exogenous, which means we are in partial equilibrium. This is appropriate if we think of the set of agents insured by this company as being small compared to the economy. Nothing important would change if we endowed agents with financial wealth and determined \(\xi\) in a competitive equilibrium. Taking \(\xi\) nonatomic means the distribution of \(\xi\) has no mass point.
density (alternatively called a stochastic discount factor or pricing kernel) is standard in finance (see, e.g. Dybvig and Ross (1997)). Using the state-price density to price insurance claims as well implicitly assumes that the private shocks underlying the claims are valued risk-neutrally. We offer two alternative motivations for this assumption leading to (1). One is a mutual insurance company’s constraint for the representative customer when the insurance company does not face actuarial risk in aggregate. To see this, note that we can write
\[ P(\xi, C) = E[P(\xi, C)\mid \xi] + (P(\xi, C) - E[P(\xi, C)\mid \xi]), \tag{2} \]
where \( E[P(\xi, C)\mid \xi] \) is the market risk and \( (P(\xi, C) - E[P(\xi, C)\mid \xi]) \) is idiosyncratic risk which sums to zero across a continuum of agents. Alternatively, it is the individual constraint given actuarially fair pricing of insurance against \( C \) conditional on \( \xi \). This can be seen by the following equation resulting from the property of iterated expectation:
\[ E[\xi E[P(\xi, C)\mid \xi]] = E[\xi P(\xi, C)]. \tag{3} \]
\( E[P(\xi, C)\mid \xi] \) is the actuarially fair price given \( \xi \), so the left-hand side of (3) gives the unconditional actuarially fair price of \( P(\xi, C) \), which by the law of iterated expectations equals the valuation in (1) and equals the right-hand side of (2) based on the state price density.

We assume an additively-separable von-Neumann-Morgenstern utility function \( U_H(H) + U_W(W) \) defined over positive \( H \) and \( W \), where \( U_H \) is utility of housing and \( U_W \) is utility of wealth. All agents have identical preferences and identical initial wealth so that each agent has expected utility:
\[ E[U_H(H - C + R(\xi, C)) + U_W(P(\xi, C) - \gamma R(\xi, C))]. \]
We will assume \( U_H \) and \( U_W \) are \( C^1 \), strictly concave, increasing, and \( U_H' \) and \( U_W' \) take on all positive real values. \( R \) is the value of the repair, and the cost of the repair is \( \gamma R \) where \( \gamma > 0 \). Both \( P \) and \( R \) are functions of \( \xi \) and \( C \), i.e. \( P : \mathbb{R}^+ \times [0, H) \to \mathbb{R} \) and \( R : \mathbb{R}^+ \times [0, H) \to \mathbb{R} \). We are assuming that the agents cannot sell their houses (or optimally do not choose to sell), perhaps because the total cost of selling the house and moving

\[^{4}\text{We describe informally the diversification across a continuum of agents, but rigorous justification can be obtained by the construction in the important paper Green (1994).}\]
elsewhere is large. Therefore, the choice of whether to repair depends on the agent’s preferences and not on some market valuation of whether the increase in house price is bigger than the cost of repair (which would complicate the analysis without further illuminating the points we are making). It is reasonable to assume that $R$ is no larger than the casualty loss, i.e. $0 \leq R \leq C$, that is, we are considering a repair, not an addition to the house ($R > C$) or selling off part of the house ($R < 0$).

The optimal payment $P$ and repair $R$ solve the following problem:

\begin{equation}
\text{Choose } P : \mathbb{R}_{++} \times [0, H) \to \mathbb{R} \text{ and } R : \mathbb{R}_{++} \times [0, H) \to \mathbb{R} \text{ to}
\end{equation}

\begin{align}
\text{maximize } & \ E[U_H(H - C + R(\xi, C)) + U_W(P(\xi, C) - \gamma R(\xi, C))] \\
\text{subject to } & \ W_0 = E[\xi P(\xi, C)] \text{ and} \\
\text{for all } & \xi \in \mathbb{R}_{++} \text{ and } C \in [0, H), \ 0 \leq R(\xi, C) \leq C.
\end{align}

The solution to the above problem is formally derived in Appendix A. With $I_W$ denoting the inverse function of $U_W'$, the optimal final wealth is given by

\begin{equation}
P^*(\xi, C) = I_W(\lambda \xi) + \gamma R(\xi, C), \tag{5}
\end{equation}

where $\lambda$ is the Lagrangian multiplier for the budget constraint. Although the final wealth is unique (and in particular this is the unique direct mechanism in which agents report their types), the optimal mechanism is not. As is usual in optimal contracting, we are free to choose any institution that implements these payoffs. Since $R(\xi, 0) = 0$ (the only feasible choice), we choose to interpret the first term in the right-hand side of (5) as the payoff to investments, i.e. $W^*(\xi) = P^*(\xi, 0)$, and the second term in (5) as the payoff on an insurance claim, i.e. $V^*(\xi, C) = P^*(\xi, C) - W^*(\xi)$. This form of the solution states that whatever the policy for repair (the $R$ function), insurance will pay for all repairs, but pays nothing if there is a casualty loss that is not repaired.$^5$ Not surprisingly, the first term in (5) is of the form we would have in a single-good world without any insurance or insurable event.

The optimal repair policy is dependent on the state realization $\xi$. Letting $I_H$ be the inverse function of the marginal utility of housing $U_H'$, optimal repair policy can be written

\begin{footnotesize}
Another institution would do all investment through the insurance company and take $V^*(\xi, C) = P^*(\xi, C)$ and $W^*(\xi) = 0$. However, the institution in the text is sensible and gives interpretable results.
\end{footnotesize}
By

\[ R^*(\xi, C) = \begin{cases} 
0 & \text{if } I_H(\gamma \lambda \xi) < H - C \\
I_H(\gamma \lambda \xi) - H + C & \text{if } H - C < I_H(\gamma \lambda \xi) < H \\
C & \text{if } I_H(\gamma \lambda \xi) > H.
\end{cases} \tag{6} \]

By the concavity of \( U_H \), \( I_H(\gamma \lambda \xi) \) is decreasing in \( \xi \), which implies that the optimal repair \( R^*(\xi, C) \) is non-increasing in \( \xi \). Note that each of “repair all” \( (R = C > 0) \) and “no repair” \( (R = 0) \) and “partial repair” \( (0 < R < C) \) is optimal for some \( \xi \). Therefore, the optimal repair policy has full repair when \( C > 0 \) in inexpensive states (low \( \xi \) states, probably when the stock market is up) and little or no repair when \( C > 0 \) in expensive states (large \( \xi \) states, probably when the stock market is down). The optimal repair policy is dependent on the state realization \( \xi \).

Given \( \lambda \), (5) and (6) provide an explicit solution to the optimal final wealth and to the optimal repair. The Lagrangian multiplier is determined endogenously through the budget constraint. The exact optimal terminal wealth and optimal repair policy depend on the law for \( \xi \), the utility function, the probability distribution of the loss, and the parameters \( H \) and \( W_0 \). The multiplier \( \lambda \) solves the budget constraint: \(^6\)

\[ W_0 = E[\xi I_W(\lambda \xi)] + E \left[ \xi \gamma \left( (I_H(\gamma \lambda \xi) - H + C)1_{H-C < I_H(\gamma \lambda \xi) < H} + C 1_{I_H(\gamma \lambda \xi) > H} \right) \right]. \tag{7} \]

The right-hand side term of (7) can be expressed as two-dimensional integrals over \( \xi \) and \( C \). Solving (7) requires then one-dimensional search for \( \lambda \). Therefore, this is easy to solve numerically in general. In some special cases, the calculation is easier. For example, if we have power utility, \( \xi \) is uniformly distributed over some intervals and \( C \) is Bernoulli, the expectations in (7) can be computed explicitly. Or, if the preferences are CRRA and the state price density \( \xi \) is assumed to be lognormal, \(^7\) the integrals over \( \xi \) reduce to applications of the Black-Scholes formula and the integrals in (7) are one-dimensional integrals. The integrals in (7) are also one-dimensional integrals when \( C \) is Bernoulli.

\(^6\)Note that when \( H - C < I_H(\gamma \lambda \xi) < H \), we must have \( C > 0 \).

\(^7\)A lognormal state price density is also consistent with the Black-Scholes-Merton model with i.i.d returns, and with many other stationary models in continuous time with Vasicek term structure (see, for example, Section 3 of Dybvig (1988) or Examples 1 and 2 of Dybvig and Rogers (1997)).
In the remainder of the section, we want to mention shortly what happens in the Bernoulli case if we assume partial repair is infeasible, so that the agent’s repair policy $R$ can be chosen to be 0 ("no repair") or $C$ ("repair all"). If only these two repair policies are available, there exists a critical value $\xi^*$ so that in less expensive states $\xi < \xi^*$, "repair all" is chosen over "no repair" and in more expensive states $\xi > \xi^*$, "no repair" is chosen. Let us now take a look at how $\xi^*$ is determined. Given $C = c$, if $R$ is chosen equal to $c$ (corresponding to "repair all" case), it results in the value

$$L_{R=c} = U_H(H) + U_W(I_W(\lambda \xi)) + \lambda(W_0 - \xi(I_W(\lambda \xi) + \gamma c)),$$

whereas if $R$ is chosen equal to 0 (corresponding to "no repair" case), we have the value

$$L_{R=0} = U_H(H - c) + U_W(I_W(\lambda \xi)) + \lambda(W_0 - \xi I_W(\lambda \xi)).$$

"Repair all" is better than "no repair" when

$$L_{R=c} > L_{R=0} \iff \xi < \xi^* \equiv \frac{1}{\lambda \gamma c} (U_H(H) - U_H(H - c)). \tag{8}$$

Condition (8) indicates that in the less expensive states, "repair all" is preferable to "no repair", whereas in the more expensive states, "no repair" will be chosen. Because the insurance pays only if there is a repair, the optimal insurance policy fully insures the casualty loss in less expensive states and does not pay off anything in more expensive states.

To summarize, under additively-separable preferences, we have shown the following: a) Given $C$ and $\xi$, optimal repair depends on the state of securities market. “Repair all” is chosen when the economy is good (when $\xi$ is small), and “no repair” is chosen when economy is bad (when $\xi$ is large). If partial repair is feasible, then there is an intermediate range of states, in which partial repair is desirable. b) The optimal insurance payment is equal to the optimal repair cost, which implies that there is no payment when there is no repair, but optimal repair is fully insured. c) Insurance does not make agents whole.

2. Efficiency comparison

This section aims to examine the degree of efficiency of our optimal insurance and repair framework. We compare

a) optimal insurance,
b) full insurance with full repair, and
c) no insurance with no repair.

We make comparisons among these cases by considering the expected utility, although for numerical work we will equivalently use the certainty equivalent (CE) defined by

\[ U_W(CE) + U_H(H) = EU, \]

which makes the quantities easier to interpret. The choice of \( H \) as the argument at \( U_H \) is arbitrary but not critical for our purpose.\(^8\) We use \( CE_{OI} \), \( CE_{FI} \) and \( CE_{NI} \) to denote the certainty equivalent achieved under optimal insurance, full insurance with full repair and no insurance with no repair.

We first explore in Table 1 a numerical example for CRRA preferences and a casualty loss which is Bernoulli distributed and drawn independently from the state price density which is lognormally distributed. \( U_H \) and \( U_W \) both have the same relative risk aversion \( \kappa \) and the parameters for the lognormal distribution of \( \xi \) are taken from Black-Scholes-Merton lognormal model with a one-year horizon instantaneous rate \( r \), mean stock return \( \mu \), and standard deviation \( \sigma \). \( C \) takes the value \( c \) with probability \( p \) and 0 with probability \( 1 - p \). More details can be found in Appendix B.

In Table 1, the certainty equivalents are computed for varying cost of repair (\( \gamma \)) for three different levels of risk aversion \( \kappa = 2, 5, 10 \). There are several observations. First, \( CE_{NI} \) does not depend on \( \gamma \) because no repair is done in the no insurance case. Second, as \( \gamma \) increases, \( CE_{OI} \) and \( CE_{FI} \) both decrease. A higher \( \gamma \) is a higher cost of repair, which makes the agent strictly worse off because for both optimal insurance and full insurance and whatever \( \gamma \), there is repair when \( \xi \) is small enough. Third, we observe that \( CE_{OI} > CE_{FI} \) and \( CE_{OI} > CE_{NI} \), because \( CE_{OI} \) is optimal and the other strategies are not, although the suboptimality may not be numerically significant. Fourth, as \( \gamma \) decreases, the difference \( CE_{OI} - CE_{FI} \) approaches 0, while the certainty equivalent for no insurance and no repair is smaller. For insurance at \( \gamma = 0.1 \), the differences are already very close to zero. The reason is that the cost of repair is going to zero and we repair almost all of the time almost

\(^8\)For some \( U_W \), using \( U_H(H) \) might be a problem because (9) might not have solutions for all values of \( EU \). However, this is never a problem in our analysis.
for free. We also have
\[
\frac{CE_{OI} - CE_{FI}}{CE_{OI} - CE_{NI}} \to 0
\]
because the optimal policy has repair almost all the time (insured because preferences are additively separable), so that full insurance is approximately optimal. Fifth, the value of \(CE_{NI}\) is higher than \(CE_{FI}\) when \(\gamma\) is large enough and smaller than \(CE_{FI}\) when \(\gamma\) is small enough. It is due to the fact that as \(\gamma\) increases, the difference \(CE_{OI} - CE_{NI}\) becomes smaller because optimal policy is to repair in fewer and fewer states. However, the full insurance policy is worse and worse because of buying more and more insurance that actually increases risk. In fact, full insurance becomes infeasible if \(\gamma\) is high enough (because the cost of insurance exceeds initial wealth). These results are marked with * in Table 1.

For the less risk-averse agent (here \(\kappa = 2\)), it seems that there are no substantial differences in the certainty equivalent between full or no insurance (the choice in practice) and buying optimal insurance. In other words, optimal insurance does not seem to result in much of an efficiency gain. For a more risk-averse agent (here \(\kappa = 5\) or \(\kappa = 10\)), these differences are much more substantial when \(\gamma\) takes some intermediate values (here \(\gamma = 1, 2, 5\)), i.e. it is important for a more risk-averse agent to buy optimal insurance when repair is expensive.

2.1. General asymptotic results for \(\gamma\) large or small. Although the numerical results are for a special case, many of the observations hold in general. Particularly, if the cost of repair becomes very small, the optimal insurance is more like full insurance with full repair. The expected utilities from these two cases approach the same limit and are larger than the expected utility for no insurance with no repair. If the cost of repair becomes extremely large, the optimal insurance is more like no insurance with no repair and the expected utilities from these two cases approach the same limit. Full insurance to implement full repair becomes infeasible as \(\gamma\) increases, because the cost of paying for full repair exceeds initial wealth. The remainder of this section shows these asymptotic results for \(\gamma\) becomes small or large. Before stating our results, we need some regularity assumptions.

**Assumption 2.1.** The riskless asset has finite price: \(E[\xi]\) is finite.
The next two assumptions are sufficient for there to exist an optimum for a portfolio problem. Often papers (see for example Cox and Huang (1989) and Dybvig and Rogers (1997)) will not make these assumptions directly, but instead there are assumptions (e.g.
growth conditions on the behavior of the distribution of $\xi$ and the utility function at 0 and $\infty$) that imply these conditions. Assumption 2.2 implies there is a solution to the first-order condition of the usual problem (without insurance) for all positive $W_0$, while Assumption 2.3 implies that the solution to the first-order condition has finite expected utility.

**Assumption 2.2.** $q(\lambda) \equiv E[\xi I_W(\lambda \xi)]$ is $C^1$ on $(0, \infty)$ and takes on all values in $(0, \infty)$. Note that strict concavity and differentiability of $U_W$ implies $I_W$ is continuous and strictly decreasing, so that $q$ is strictly decreasing. Therefore $q^{-1}$ is $C^1$ on $(0, \infty)$ and takes on all values in $(0, \infty)$.

**Assumption 2.3.** For all $\lambda > 0$, $E[U_W(I_W(\lambda \xi))]$ is finite.

Our next assumption says that there is some minimum amount of value left in the house. This avoids some economically uninteresting technical issues.

**Assumption 2.4.** There exists $\delta > 0$ such that $P(H - C > \delta) = 1$.

Finally, we rule out the trivial case in which an insurable event can never happen.

**Assumption 2.5.** $P(C > 0) > 0$.

**Theorem 2.6.** As $\gamma \to 0$, expected utility for full insurance with full repair and expected utility for the optimal contract both approach the same limit, while expected utility for no insurance and no repair is smaller. As $\gamma \uparrow \infty$, expected utility for no insurance and no repair and expected utility for the optimal contract both approach the same limit, while full insurance with full repair becomes infeasible.

**Proof:** Convergence of some expectations used in this proof follows from dominated convergence and is given by Lemmas 4.1 and 4.2 in Appendix C. Denote by $\lambda_\gamma$, the Lagrange multiplier for this problem, indexed by $\gamma$ which is the parameter of interest. This multiplier satisfies the budget constraint,

$$E[\xi I_W(\lambda_\gamma \xi)] + E[\xi \gamma R] = W_0, \quad (10)$$

where $R$ is the optimal policy given gamma.
Consider first the optimal contract in the case $\gamma \downarrow 0$. The first term in (10) is $q(\lambda, \gamma)$ where $q$ is the invertible bi-continuous function defined in Assumption 2.2. As shown by Lemma 4.1 in Appendix C, $\lim_{\gamma \to 0} E[\xi \gamma R] = 0$ because $\gamma \downarrow 0$ and other factors are bounded in expectations. Therefore, $\lim_{\gamma \to 0} \lambda = \lim_{\gamma \to 0} q^{-1}(W_0 - E[\xi \gamma R]) = \lambda_0$, where $\lambda_0 = q^{-1}(W_0)$ is the Lagrangian multiplier for the problem with zero cost.

Given $\gamma$, the expected utility of the optimal contract is given by

$$E[U_W(I_W(\lambda, \gamma, \xi)) + U_H(\min(H, \max(I_H(\gamma, \lambda, \gamma, \xi), H - C)))].$$

Lemma 4.2 in Appendix C shows that the expected utility of optimal strategy converges to $E[U_W(I_W(\lambda_0, \xi)) + U_H(H)]$ as $\gamma \to 0$.

Now consider buying full insurance to implement full repair. This is feasible when $\gamma$ is small enough, specifically when $\gamma E[\xi C] < W_0$ so the agent can afford the insurance and still have positive wealth left over to invest and consume. Assuming the conditionally optimal portfolio strategy, the budget constraint in this case is

$$E[\xi I_W(\lambda^{fi}_\gamma, \xi)] + E[\xi \gamma C] = W_0,$$

where the full insurance Lagrange multiplier $\lambda^{fi}_\gamma$ is determined by the budget constraint. By the same argument as for the optimal contract, we have that $\lambda^{fi}_\gamma \to \lambda_0$ as $\gamma \to 0$. The expected utility of this strategy of buying full insurance to implement full repair is

$$E[U_W(I_W(\lambda^{fi}_\gamma, \xi)) + U_H(H)].$$

We have just shown that $\lambda^{fi}_\gamma \to \lambda_0$ as $\gamma \to 0$, and continuity of the first term in $\lambda^{fi}_\gamma$ follows from Assumption 2.3. Therefore, as $\gamma \to 0$, the expected utility of full insurance with full repair tends to $E[U_W(I_W(\lambda_0, \xi)) + U_H(H)]$, which is the same as the limiting value of the optimal contract.

Now consider having no insurance and never repairing. Assuming the conditionally optimal strategy for investing, the budget constraint is

$$E[\xi I_W(\lambda^{ni}_\gamma)] + E[\xi \gamma R] = W_0,$$
where $R = 0$ and therefore $\lambda_{ni}^\gamma = \lambda_0$. Then the expected utility is given by

$$E[U_W(I_W(\lambda_0 \xi)) + U_H(H - C)],$$

which does not depend on $\gamma$ since there is no repair. Therefore, the limit of the expected utility as $\gamma \to 0$ is $E[U_W(I_W(\lambda_0 \xi))] + E[U_H(H - C)]$, which differs from the asymptotic value for optimal insurance which has the term $U_H(H)$ in place of $E[U_H(H - C)]$. Since $C$ is nonnegative and $P(C > 0) > 0$ (Assumption 2.5), as $\gamma \to 0$, the expected utility for no insurance and no repair tends to a smaller limit than expected utility for optimal insurance.

Now let us consider the limiting case $\gamma \to \infty$. As shown by Lemma 4.1 in Appendix C, the second term in the budget constraint $E[\xi \gamma R]$ tends to zero as $\gamma \to \infty$. For the optimal insurance case, recall the value function:

$$E[U_W(I_W(\lambda_{ni}^\gamma \xi)) + U_H(H - C + R_{\gamma})].$$

(11)

$\lambda_{\gamma}^\gamma \to \lambda_0$ implies that the first term in (11) converges to $E[U_W(I_W(\lambda_0 \xi))]$ as $\gamma \uparrow \infty$. Lemma 4.2 in Appendix C shows that the second term converges to $E[U_H(H - C)]$. Therefore, expected utility for the optimal contract converges to $E[U_W(I_W(\lambda_{ni}^\gamma \xi)) + U_H(H - C)]$. As derived earlier, this is the same as the expected utility of no insurance with no repair for all $\gamma$, and therefore in the limit as $\gamma \to \infty$. Full insurance with full repair has a cost of insurance

$$E[\xi \gamma C] = \gamma E[\xi C] \xrightarrow{\gamma \to \infty} \infty$$

since $\xi > 0$ and $C \geq 0$ with positive probability. Therefore, for $\gamma$ large enough, the cost of insurance exceeds $W_0$ and cannot be consistent with budget constraint. □

3. Non-additively-separable preferences

In this section, the assumption of additively-separable preferences made in the benchmark model is relaxed to examine the robustness of the three main results. We discuss several specific utility functions and also provide some results for general preferences. Throughout we assume partial repair is feasible (as in our benchmark case).

Section 3.1 deals with the case in which housing and other consumption are perfect substitutes, i.e., are treated as a single good. Given perfect substitutes, the optimal insurance policy has full insurance and the agent is always made whole. In this case alone,
optimal insurance and repair policy does not depend on the financial aggregates. Section 3.2 discusses the case in which housing and other consumption are perfect complements, i.e. utility is a function of the maximum of consumption and a multiple of housing. Given perfect complements, the optimal insurance has partial insurance and is not equal to the cost of repair. It holds that optimal insurance and optimal repair depend on the security market conditions. Sections 3.3 and 3.4 analyze two cases in which we have the same static preferences for H and W as for log utilities but different risk preferences: \( U(H, W) = -\frac{1}{HW} \) and \( U(H, W) = H^{1/3}W^{1/3} \). In the former case, we obtain similar results as under perfect complements. In the latter case, we obtain a counterintuitive result. When there is a casualty loss and the agent chooses not to repair, the optimal insurance is negative. We interpret this as consistent with the very low relative risk aversion of these preferences. Subsection 3.4.1 provides some insight into a general non-additively-separable preferences, for which optimal full repair is fully insured.

3.1. **Perfect substitutes:** \( U(aH + W), a > 0 \). Housing and wealth are perfect substitutes. In this case, the agent faces the following optimization problem:

Choose \( P : \mathbb{R}_{++} \times [0, H) \to \mathbb{R} \) and \( R : \mathbb{R}_{++} \times [0, H) \to \mathbb{R} \) to

\[
\text{maximize} \quad E[U(a(H - C + R(\xi, C)) + P(\xi, C) - \gamma R(\xi, C))] \\
\text{subject to} \quad W_0 = E[\xi P(\xi, C)] \quad \text{and} \\
\quad \text{for all} \quad \xi \in \mathbb{R}_{++}, \ C \in [0, H), \ 0 \leq R(\xi, C) \leq C.
\]

We assume \( U \) is \( C^1 \), strictly concave, and increasing. The optimal solutions can be derived similarly as in Appendix A. Here we ignore the details and jump to the solution immediately. Letting \( I \) be the inverse function of \( U' \), we can express the optimal terminal wealth under full insurance as follows:

\[
P^*(\xi, C) = I(\lambda \xi) - aH + (\gamma - a)R(\xi, C) + aC.
\]

Substituting this back into the utility function, we obtain a utility of \( U(I(\lambda \xi)) \) which depends on \( \xi \), but not on \( C \). It implies that the optimal insurance policy is full insurance in the sense that given the state \( \xi \) of the market the insurance makes the agent indifferent about whether there is a casualty loss or not.
In this case, optimal repair when $C > 0$ depends on the parameter $a$ but not on the market state $\xi$:

$$R^*(\xi, C) = \begin{cases} 
0 & \text{if } a < \gamma \\
[0, C] & \text{if } a = \gamma \\
C & \text{if } a > \gamma.
\end{cases}$$

If $C = 0$, the insurance pays off nothing and the optimal terminal wealth corresponds to $I(\lambda \xi) - a H$, and no repair is needed. If $a > \gamma$, full repair is always chosen, whereas if $a < \gamma$, no repair is always chosen. If $a = \gamma$, any feasible repair policy is optimal.

It is a quite intuitive result: since house and monetary wealth are perfect substitutes, a dollar’s worth of repair costs $\gamma$ dollars and always has a subjective benefit worth $a$ dollar. If the benefit exceeds the cost ($a > \gamma$), repair is always optimal, whereas if cost exceeds the benefit ($\gamma > a$), repair is never optimal. Insurance follows from the traditional result for the single-good case absent frictions and information asymmetries, and always makes the agent whole whether or not repair is optimal. The only difference with repair is that when repair is preferred ($a > \gamma$), the possibility of repair reduces the cost of making the agent whole.

3.2. **Perfect complements.** Suppose that the agent’s utility is characterized by $U(H, W) = \frac{(\min\{H, aW\})^{1-\rho}}{1-\rho}$. The interesting case has $\lambda > 0$, i.e. the budget constraint is strictly binding. The agent is satiated if all losses are repaired ($R(\xi, C) \equiv C$) and consumption is at least $H/a$. Therefore, if $W_0 \geq E[\xi(H/a + \gamma C)]$, then it is feasible to achieve the global optimum (conditional on $H$) and $\lambda = 0$. The choice problem under perfect complements

---

9If $a = 1$, the agent is indifferent between repairing and not repairing, and the agent could choose a repair policy that depends on the market state. However, this policy would be no better than choosing a policy that does not depend on the market state.
is as follows:

Choose \( P : \mathbb{R}_{++} \times [0, H) \rightarrow \mathbb{R} \) and \( R : \mathbb{R}_{++} \times [0, H) \rightarrow \mathbb{R} \) to maximize

\[
E \left[ \frac{1}{1 - \rho} (\min \{ H - C + R(\xi, C), a(P(\xi, C) - \gamma R(\xi, C)) \})^{1 - \rho} \right]
\]

subject to \( W_0 = E[\xi P(\xi, C)] \) and

\[
\text{for all } \xi \in \mathbb{R}_{++}, \text{ and } C \in [0, H), \ 0 \leq R(\xi, C) \leq C.
\]

Let \( K(\xi, C) \equiv P(\xi, C) - \gamma R(\xi, C) \), which is consumption, i.e., the wealth after repair. It is useful to reparameterize the problem as a choice of \( K(\xi, C) \) and \( R(\xi, C) \) instead of \( P(\xi, C) \) and \( R(\xi, C) \). The minimum in the objective function of (13) implies we can infer a lot about the solution. For example, there is no use choosing \( R > 0 \) when \( aK(\xi, C) < H - C \) and no use choosing \( aK(\xi, C) > H \), since we would be paying a cost (positive since \( \lambda > 0 \)) for no benefit. Also, when there is partial repair, we must have \( H - C - R(\xi, C) = aK(\xi, C) \) or else at least part of the cost of consumption or repair would be wasted. This implies w.l.o.g that \( R(\xi, C) = \min \{ C, \max \{ 0, aK(\xi, C) - (H - C) \} \} \). First ignoring the constraint on the repair policy and substituting the definition of \( K(\xi, C) \) and this expression for \( R(\xi, C) \) into the integrand of the Lagrangian, we have that

\[
L = \begin{cases} 
\frac{(aK(\xi, C))^{1 - \rho}}{1 - \rho} + \lambda (W_0 - \xi K(\xi, C)) & \text{if } K < \frac{H-C}{a} \\
\frac{(aK(\xi, C))^{1 - \rho}}{1 - \rho} + \lambda (W_0 - \xi (K(\xi, C) + \gamma(aK(\xi, C) - (H - C)))) & \text{if } \frac{H-C}{a} < K < \frac{H}{a} \\
\frac{(H)^{1 - \rho}}{1 - \rho} + \lambda (W_0 - \xi (K(\xi, C) + \gamma C)) & \text{if } K > \frac{H}{a}
\end{cases}
\]

and the gradient is

\[
\frac{\partial L}{\partial K} = \begin{cases} 
\frac{a^{1 - \rho} (K(\xi, C))^{-\rho} - \lambda \xi}{a^{1 - \rho} (K(\xi, C))^{-\rho} - \lambda \xi(1 + \gamma a)} & \text{if } K < \frac{H-C}{a} \\
-\lambda \xi & \text{if } \frac{H-C}{a} < K < \frac{H}{a} \\
-\lambda \xi & \text{if } K > \frac{H}{a}.
\end{cases}
\]

Letting \( \frac{\partial L}{\partial K} = 0 \) leads to the optimal

\[
K^*(\xi, C) = \begin{cases} 
(a^{\rho-1} \lambda \xi)^{-\frac{1}{\rho}} & \text{if } K < \frac{H-C}{a} \\
(a^{\rho-1} \lambda \xi(1 + \gamma a))^{-\frac{1}{\rho}} & \text{if } \frac{H-C}{a} < K < \frac{H}{a} \\
\frac{H}{a} & \text{if } K > \frac{H}{a}.
\end{cases}
\]
State and casualty realization | $R^*(\xi, C)$ | $W^*(\xi)$ | $V^*(\xi, C)$
--- | --- | --- | ---
$C = 0$ | 0 | $(a^{\rho-1}\lambda\xi)^{-\frac{1}{\rho}}$ | 0
$C > 0$ and $\xi > \frac{a(H-C)^{-\rho}}{(1+\lambda a)}$ | 0 | $(a^{\rho-1}\lambda\xi)^{-\frac{1}{\rho}}$ | 0
$C > 0$ and $\frac{a(H-C)^{-\rho}}{(1+\lambda a)} > \xi > \frac{a(H)^{-\rho}}{(1+\lambda a)}$ | $a(a^{\rho-1}\lambda(1 + \gamma a))^{-\frac{1}{\rho}} - (H - C)$ | $(a^{\rho-1}\lambda\xi)^{-\frac{1}{\rho}}$ | $(1 + \gamma a)^{1-\frac{1}{\rho}} - 1)(a^{\rho-1}\lambda\xi)^{-\frac{1}{\rho}}$
$C > 0$ and $\xi < \frac{a(H)^{-\rho}}{(1+\lambda a)}$ | $C$ | $(a^{\rho-1}\lambda\xi)^{-\frac{1}{\rho}} - \gamma C$

Table 2. Optimal repair policy $R^*$, portfolio payoff $W^*$, and insurance payoff $V^* = P^*-W^*$ for perfect complements with $U(H, W) = \frac{(\min(H, aW))^{1-\rho}}{1-\rho}$ as a function of the casualty loss $C$ and state price density $\xi$.

Note that the middle region is empty when $C = 0$. The Lagrangian is concave (strictly concave in the relevant region $K < H/a$) and not differentiable at the break points $(H - C)/a$ and $H/a$. At these points, the derivatives correspondence (subgradient) is the closed interval from the right derivative to the left derivative. The maximum is the value of $K$ for which the derivative or an element of the subgradient is 0.

Substituting the optimal $K^*(\xi, C)$ back to the optimal repair policy when $C > 0$, we have:

$$R^*(\xi, C) = \begin{cases} 
0 & \text{if } \xi > \frac{a(H-C)^{-\rho}}{(1+\lambda a)} \\
(a^{\rho-1}\lambda(1 + \gamma a))^{-\frac{1}{\rho}} - (H - C) & \text{if } \frac{a(H-C)^{-\rho}}{(1+\lambda a)} > \xi > \frac{a(H)^{-\rho}}{(1+\lambda a)} \\
C & \text{if } \xi < \frac{a(H)^{-\rho}}{(1+\lambda a)}.
\end{cases}$$

We obtain the optimal terminal wealth under perfect complements as follows:

$$P^*(\xi, C) = \begin{cases} 
(a^{\rho-1}\lambda\xi)^{-\frac{1}{\rho}} & \text{if } \xi > \frac{a(H-C)^{-\rho}}{(1+\lambda a)} \\
(1 + \gamma a)(a^{\rho-1}\lambda(1 + \gamma a))^{-\frac{1}{\rho}} - \gamma(H - C) & \text{if } \frac{a(H-C)^{-\rho}}{(1+\lambda a)} > \xi > \frac{a(H)^{-\rho}}{(1+\lambda a)} \\
(a^{\rho-1}\lambda\xi)^{-\frac{1}{\rho}} + \gamma C & \text{if } \xi < \frac{a(H)^{-\rho}}{(1+\lambda a)}.
\end{cases}$$

The optimal terminal wealth can be decomposed into the optimal investment and optimal insurance as in Table 2. Under perfect complements, optimal repair still depends on security market conditions, with full repair in inexpensive states and little or no repair in expensive states. The agent is not made whole, since the loss is fully compensated only when it is fully repaired.
3.3. \( U(H, W) = -\frac{1}{H W} \). In this case we have the same static preferences for \( H \) and \( W \) as for log utility, but different risk preferences. We leave the detailed derivation for the optimal solutions to the reader and describe immediately the solutions.

The optimal final wealth is given

\[
P^*(\xi, C) = \left( \frac{1}{\lambda \xi H - C + R(\xi, C)} \right)^{\frac{1}{2}} + \gamma R(\xi, C). \tag{14}
\]

Substituting this back into the utility function, we obtain a utility as a function of both \( \xi \) and \( C \):

\[
- (\lambda \xi)^{\frac{1}{2}} (H - C + R(\xi, C))^{-\frac{1}{2}},
\]

which depends on whether there is a casualty loss if there is not full repair. In other words, the agent is not made whole when there is a casualty loss and there is not full repair. In fact, for some \( \xi \), partial insurance is desirable.

We have the following optimal repair policy when \( C > 0 \)

\[
R^*(\xi, C) = \begin{cases} 
0 & \text{if } \xi > \frac{1}{\lambda \gamma} (H - C)^{-3} \\
\gamma^{-2/3}(\lambda \xi)^{-1/3} - H + C & \text{if } \frac{1}{\lambda \gamma} H^{-3} < \xi < \frac{1}{\lambda \gamma} (H - C)^{-3} \\
C & \text{if } \xi < \frac{1}{\lambda \gamma} H^{-3}.
\end{cases}
\]

Substituting the optimal repair policy back to the optimal terminal wealth leads to

\[
P^*(\xi, C) = \begin{cases} 
\left( \frac{1}{\lambda \xi (H - C)} \right)^{1/2} & \text{if } \xi > \frac{1}{\lambda \gamma} (H - C)^{-3} \\
2\gamma^{2/3}(\lambda \xi)^{-1/3} - H + C & \text{if } \frac{1}{\lambda \gamma} H^{-3} < \xi < \frac{1}{\lambda \gamma} (H - C)^{-3} \\
\left( \frac{1}{\lambda \xi H} \right)^{1/2} + \gamma C & \text{if } \xi < \frac{1}{\lambda \gamma} H^{-3}.
\end{cases}
\]

In Table 3 the optimal terminal wealth is decomposed to two parts: optimal investment \( W^*(\xi) \) in the financial market and optimal insurance \( V^*(\xi, C) \). It is observed that partial insurance might be desirable, depending on the optimal repair policy which is further dependent on the realization of \( \xi \) and \( C \). It does not hold that optimal repair is equal to optimal insurance, but it is true that full repair is fully insured.

3.4. \( U(H, W) = H^{1/3} W^{1/3} \). Again the static preferences are equivalent to log utility, but relative risk aversion over wealth is less than 1. The optimal terminal wealth is given by

\[
P^*(\xi, C) = \frac{(H - C + R(\xi, C))^{1/2}}{(3\lambda \xi)^{3/2}} + \gamma R(\xi, C). \tag{15}
\]
State and casualty realization | $R^*(\xi, C)$ | $W^*(\xi)$ | $V^*(\xi, C)$
---|---|---|---
$C = 0$ | 0 | $\left(\frac{1}{\lambda H}\right)^{1/2}$ | 0
$C > 0$ and $\xi > \frac{1}{\lambda^2}(H - C)^{-3}$ | 0 | $\left(\frac{1}{\lambda H}\right)^{1/2}$ | $\left(\frac{1}{\lambda H} - \left(\frac{1}{\lambda H}\right)^{1/2}\right)$
$C > 0$ and $\frac{1}{\lambda^2}H^{-3} < \xi < \frac{1}{\lambda^2}(H - C)^{-3}$ | $\gamma^{-2/3}(\lambda\xi)^{-1/3}$ | $\left(\frac{1}{\lambda H}\right)^{1/2}$ | $2\gamma^{2/3}(\lambda\xi)^{-1/3}$
$C > 0$ and $\xi < \frac{1}{\lambda^2}H^{-3}$ | $\left(\frac{1}{\lambda H}\right)^{1/2}$ | $\gamma C$

**Table 3.** Optimal repair policy $R^*$, portfolio payoff $W^*$, and insurance payoff $V^* = P^* - W^*$ for $U(H, W) = -\frac{1}{HW}$ as a function of the casualty loss $C$ and state price density $\xi$.

Plugging this back in the utility function, we have a utility of

$$\frac{(H - C + R(\xi, C))^{3/2}}{3\lambda\xi},$$

which depends on $\xi$ and casualty loss $C$. It means that full insurance is not an optimal insurance policy. The agent’s utility is not equalized in all circumstances, particularly for the two scenarios: if there is no casualty loss; and if there is casualty loss and “no repair” is chosen. If there is casualty loss but no repair, the agent is not made whole.

The optimal repair policy when $C > 0$ is given by

$$R^*(\xi, C) = \begin{cases} 0 & \text{if } \xi > \frac{1}{3\lambda} \left(\frac{1}{(H - C)\gamma}\right)^{3/2} \\ \gamma^{-2/3}(\lambda\xi)^{-1/3} - H + C & \text{if } \frac{1}{3\lambda} \left(\frac{1}{H\gamma^2}\right)^{3/2} < \xi < \frac{1}{3\lambda} \left(\frac{1}{(H - C)\gamma}\right)^{3/2} \\ C & \text{if } \xi < \frac{1}{3\lambda} \left(\frac{1}{H\gamma^2}\right)^{3/2}. \end{cases}$$

Substituting the optimal repair policy back in $P^*(\xi, C)$, we obtain

$$P^*(\xi, C) = \begin{cases} (H - C)^{1/2} & \text{if } \xi > \frac{1}{\lambda^2}(H - C)^{-3} \\ 2\gamma^{-1}(3\lambda\xi)^{-3} - \gamma(H - C) & \text{if } \frac{1}{\lambda^2}H^{-3} < \xi < \frac{1}{\lambda^2}(H - C)^{-3} \\ \frac{(H)^{1/2}}{(3\lambda\xi)^{3/2}} + \gamma C & \text{if } \xi < \frac{1}{\lambda^2}H^{-3}. \end{cases}$$

In Table 4 the optimal terminal wealth is split again into two parts: optimal investment $W^*(\xi)$ from the securities market and optimal insurance $V^*(\xi, C)$. Very interestingly, we obtain a quite counterintuitive result. If the agent chooses not to repair at all when there
State and casualty realization | $R^*(\xi, C)$ | $W^*(\xi)$ | $V^*(\xi, C)$
--- | --- | --- | ---
$C = 0$ | 0 | $\frac{H^{1/2}}{(3\xi)^{3/2}}$ | 0
$C > 0$ and $\xi > \frac{1}{\sqrt{3\lambda}}(H - C)^{-3}$ | 0 | $\frac{H^{1/2}}{(3\xi)^{3/2}}$ | $(H-C)^{1/2} - \frac{(H)^{1/2}}{(3\xi)^{3/2}} < 0$
$C > 0$ and $\frac{1}{\sqrt{3\lambda}}H^{-3} < \xi < \frac{1}{\sqrt{3\lambda}}(H - C)^{-3}$ | $\gamma^{-2}(3\lambda\xi)^{-3}$ | $\frac{H^{1/2}}{(3\xi)^{3/2}}$ | $2\gamma^{-1}(3\lambda\xi)^{-3} - \gamma(H - C)$
$C > 0$ and $\xi < \frac{1}{\sqrt{3\lambda}}H^{-3}$ | $C$ | $\frac{H^{1/2}}{(3\xi)^{3/2}}$ | $-\frac{(H)^{1/2}}{(3\lambda\xi)^{3/2}}$

Table 4. Optimal repair policy $R^*$, portfolio payoff $W^*$, and insurance payoff $V^* = P^* - W^*$ for $H^{1/3}W^{1/3}$ as a function of the casualty loss $C$ and state price density $\xi$.

is a casualty, the optimal insurance is negative, i.e. the insurance is not going to pay off anything for the casualty and furthermore ask the agent to provide it a certain amount. It is an anti-insurance scenario and results from the fact that the utility here leads to an extremely low relative relation aversion coefficient (RRA < 1). So there is a reporting issue, i.e. unlike in all the other cases we consider, the agent actually has an incentive to hide the loss.

3.4.1. General utility. Some of the results are available for a general (not necessarily additively-separable) utility function $U(H, W)$, which is assumed to be twice differentiable, strictly monotone and strictly concave. Specifically, $U_i > 0$, $i = 1, 2$ and $(U_{ii})$ is negative definite, i.e. $U_{11} < 0$, $U_{22} < 0$ and $U_{11}U_{22} - (U_{12})^2 < 0$. Under these assumptions, the optimal payment $P$ and repair $R$ solve the following problem:

Choose $P : \mathbb{R}_{++} \times [0, H) \rightarrow \mathbb{R}$ and $R : \mathbb{R}_{++} \times [0, H) \rightarrow \mathbb{R}$ to

\[
\max E[U(H - C + R(\xi, C), P(\xi, C) - \gamma R(\xi, C))]
\]

subject to

\[
W_0 = E[\xi P(\xi, C)] \text{ and } \\
\text{for all } \xi \in \mathbb{R}_{++}, \text{ and } C \in [0, H), \ 0 \leq R(\xi, C) \leq C.
\]

Since there is no feasible repair necessary when there is no casualty loss, for $C = 0$, the associated Lagrangian function is

\[
\mathcal{L}_{C=0} = E\left[U(H, P(\xi, 0)) + \lambda(W_0 - \xi P(\xi, 0))\right].
\]
Maximizing over the integrand of the Lagrangian leads to

$$U_2(H, P(\xi, 0)) = \lambda \xi.$$  \hfill (17)

For $C > 0$, the associated Lagrangian function of the reduced problem is

$$\mathcal{L}_{C > 0} = E\left[U(H - C + R(\xi, C), P(\xi, C) - \gamma R(\xi, C)) + \lambda(W_0 - \xi P(\xi, C)) + \mu_1 R(\xi, C) + \mu_2 (C - R(\xi, C))\right]$$  \hfill (18)

is maximized over $R(\xi, C) \in [0, C]$. The solution to the optimal terminal wealth and the optimal repair policy is characterized as follows:

$$U_2(H - C + R(\xi, C), P(\xi, C) - \gamma R(\xi, C)) = \lambda \xi,$$

$$U_1(H - C + R(\xi, C), P(\xi, C) - \gamma R(\xi, C))$$

$$= \gamma U_2\left(H - C + R(\xi, C), P(\xi, C) - \gamma R(\xi, C)\right) + \mu_2 - \mu_1,$$

where $\mu_1 \geq 0$, $\mu_2 \geq 0$, $\mu_1 R(\xi, C) = 0$ and $\mu_2 (C - R(\xi, C)) = 0$. As seen in the specific examples, the optimal repair policy can have full repair, partial repair or no repair.

In the following, let us examine whether optimal repair is fully insured, i.e. whether

$$V(\xi, C) = P(\xi, C) - P(\xi, 0) = \gamma R(\xi, C).$$  \hfill (19)

Note that

$$\lambda \xi = U_2(H - C + R(\xi, C), P(\xi, C) - \gamma R(\xi, C)) = U_2(H, P(\xi, 0)).$$  \hfill (20)

For $R(\xi, C) = C$ (full repair),

$$U_2(H, P(\xi, C) - \gamma C) = U_2(H, P(\xi, 0)).$$  \hfill (21)

Since $U$ is assumed to be strictly monotone and strictly concave, (21) leads to

$$V(\xi, C) = \gamma R(\xi, C) = \gamma C = P(\xi, C) - P(\xi, 0).$$

It indicates that full repair is always fully insured. When there is partial repair, let us first assume that (19) holds. Under this assumption, (20) can be rewritten to:

$$\lambda \xi = U_2(H - C + R(\xi, C), P(\xi, 0)) = U_2(H, P(\xi, 0)).$$  \hfill (22)
Certainly, (22) does not hold generally. It does hold under additively-separable preferences, where $U_2$ does not depend on the first argument.\textsuperscript{10} Therefore, under the general utility, optimal repair is not always fully insured, but full repair is fully insured.

Another interesting question is whether the agent is made whole. In particular, we want to examine whether the agent is made full when there is not full repair. In other words, do we obtain the same utility in these two cases: “$C = 0$” and “$C > 0$, $R = 0$”? From the first order conditions, we know that

$$U_2(H, P(\xi, 0)) = U_2(H - C, P(\xi, C)) = \lambda \xi.$$  

Does it imply

$$U(H, P(\xi, 0)) = U(H - C, P(\xi, C))? \quad (23)$$

The equality in (23) does not hold in general, but it is true for perfect substitutes (analyzed earlier) and we cannot rule out that it can happen by accident.

To summarize, under general (possibly non-additive) preferences, we find: a) Except in the degenerate case of perfect substitutes, optimal repair depends on the state of the securities market. b) When full repair is optimal, it is fully insured, but otherwise the insurance payment may not equal the repair cost. c) In our examples, insurance does not make agents whole except in the degenerate case of perfect substitutes.

4. Conclusion

We study optimal insurance and repair of casualty loss in the presence of a securities market. We show that under additively-separable preferences, optimal repair depends on security market conditions; the optimal insurance payment equals the cost of optimal repair; and the agent is not made whole. Under more general (possibly non-additively-separable preferences), optimal repair still depends on the state of the securities market.

\textsuperscript{10}There is probably a sense in which this is necessary as well if (22) holds for a rich enough set of problems, since $f_2(x, y)$ is independent of $x$ for all $y$ iff $f$ is additively separable.
and insurance does not make agents whole (except in the degenerate case of perfect substitutes). Quite generally, when full repair is optimal it is fully insured.

One striking feature of the model is that the optimal insurance payoff depends both on the size of the loss and on the state of the economy. Since insurance contracts typically do not depend on economic aggregates, this presents a puzzle. It would be useful to come up with a model and/or empirical analysis to resolve this puzzle. Our analysis is a good start at solving the puzzle. From the theoretical and numerical analysis in the paper, having the optimal state-dependent contract is most important when the cost of repair is intermediate. If the repair cost is very low, having full insurance and always repairing is nearly optimal, while if the repair cost is very high, having no insurance and never repairing is nearly optimal. Not surprisingly, having the optimal insurance contract is most important when risk aversion is high.

The model predicts that under additively-separable preferences insurance contracts will only pay off on optimal repairs and do not provide any compensation for damage that is not to be repaired, and indeed some insurance contracts will pay for repairs but not provide cash directly. To the extent that insurance does pay off even when repairs are not made this might be evidence of non-separable preferences such as those analyzed in Section 3. It is an interesting question whether contracts that pay off in cash do so knowing that the beneficiary has a strong incentive to make a repair, or whether there is a useful function served by paying off in cash that is economically different than if the insurer paid for the repairs directly. For example, in a setting with asymmetric information, the second-best contract may pay off when there are no repairs because the insurance company does not have enough information to know whether repairs are optimal. In this case, the efficiency gain from allowing the beneficiary to opt out of the repair may dominate any bad incentive effects of giving the beneficiary an option.

Modifying the model with assumptions appropriate for specific insurance contracts might lead to new insights and sharper predictions. More generally, the optimal contracting framework in the paper could be used to analyze other situations, such as the choice of an optimal mortgage, in which agents face both public and private risk.
4.1. **Appendix A.** This appendix provides solutions to the problem stated in Equation (4). Given \( \xi \) and \( C \), the solution \((P^*(\xi, C), R^*(\xi, C))\) to this problem satisfies the following first order conditions with respect to \( P \) and \( R \) and complementary slackness conditions

\[
\lambda \xi = U'_W(P(\xi, C) - \gamma R(\xi, C))
\]

\[
0 = U'_H(H - C + R(\xi, C)) + U'_W(P(\xi, C) - \gamma R(\xi, C))(-\gamma) + \lambda_2 - \lambda_3
\]

\[
0 = \lambda(W_0 - E[\xi P(\xi, C)])
\]

\[
0 = \lambda_2 R(\xi, C)
\]

\[
0 = \lambda_3(C - R(\xi, C))
\]

\[
\lambda \geq 0, \lambda_i \geq 0, \ i = 2, 3.
\]

Assume that \( I_W \) and \( I_H \) are the inverse function of \( U'_W \) and \( U'_H \). The following three solution sets are possible.

- **Interior solution of** \( R(\xi, C) \in (0, C) \):

  \[
  \begin{cases}
  R(\xi, C) = C - H + I_H(\gamma \lambda_1 \xi), & P(\xi, C) = \gamma R(\xi, C) + I_W(\lambda \xi), \\
  \lambda > 0, \lambda_2 = 0, \lambda_3 = 0, & E[\xi P(\xi, C)] = W_0
  \end{cases}
  \]

- **Corner solution** \( R(\xi, C) = C \)

  \[
  \begin{cases}
  R(\xi, C) = C, & P(\xi, C) = \gamma C + I_W(\lambda_1 \xi), \\
  \lambda > 0, \lambda_2 = 0, \lambda_3 > 0, & U'_H(H) = \gamma \lambda_1 \xi + \lambda_3, \ E[\xi P(\xi, C)] = W_0
  \end{cases}
  \]

- **Corner solution** \( R(\xi, C) = 0 \)

  \[
  \begin{cases}
  R(\xi, C) = 0, & P(\xi, C) = I_W(\lambda_1 \xi), \\
  \lambda > 0, \lambda_2 > 0, \lambda_3 = 0, & U'_H(H - C) = \gamma \lambda_1 \xi - \lambda_2, \ E[\xi P(\xi, C)] = W_0
  \end{cases}
  \]

To sum up, the optimal final wealth is given by

\[
P^*(\xi, C) = I_W(\lambda \xi) + \gamma R^*(\xi, C).
\]
The optimal repair policy suggests “full repair”, “no repair” or “partial repair”, dependent of the state realizations. Note that “full repair” is chosen in those states \( \xi \) satisfying
\[
U'_H(H) = \gamma \lambda \xi + \lambda_3, \quad \text{with } \lambda, \lambda_3 > 0.
\]

“No repair” is chosen in those states with
\[
U'_H(H - C) = \gamma \lambda \xi - \lambda_2 \quad \text{with } \lambda, \lambda_2 > 0.
\]

Finally, an interior solution of \( R \in (0, C) \) is obtained when
\[
U'_H(H - C + R) = \gamma \lambda \xi \quad \text{with } \lambda > 0.
\]

Since \( U'_H(.) \) is decreasing in its argument, we have \( U'_H(H) < U'_H(H - C + R) < U'_H(H - C) \).

Assume further that the state price density process is lognormal with
\[
\ln \xi \sim N \left( - \left( r + \frac{1}{2} \eta^2 \right), \eta^2 \right), \quad (24)
\]
where \( r \) is the continuously-compounded yield on a one-period bond and \( \eta = \frac{\mu - \gamma}{\sigma} \) is the market price of risk. The assumption of lognormally distributed state price process \( \xi \)
implies that the budget constraint in (7) can be reformulated as:

\[
W_0 = \lambda^{-1/\kappa} \exp \left\{ - \left( 1 - \frac{1}{\kappa} \right) (r + \frac{1}{2} \eta^2) + \frac{1}{2} \left( 1 - \frac{1}{\kappa} \right)^2 \eta^2 \right\} + \gamma p e^{-r} N \left( \frac{\ln \frac{\nu H - c}{\lambda \gamma} + (r + \frac{1}{2} \eta^2)}{\eta} \right) \\
+ \gamma \left( \frac{\lambda}{\eta} \right)^{-1/\kappa} \exp \left\{ - \left( 1 - \frac{1}{\kappa} \right) (r + \frac{1}{2} \eta^2) + \frac{1}{2} \left( 1 - \frac{1}{\kappa} \right)^2 \eta^2 \right\} \\
\cdot \left( N \left( \frac{\ln \frac{\nu (H-c)^-}{\lambda \gamma} + (r + \frac{1}{2} \eta^2)}{\eta} - \left( 1 - \frac{1}{\kappa} \right) \eta \right) - N \left( \frac{\ln \frac{\nu (H)^-}{\lambda \gamma} + (r + \frac{1}{2} \eta^2)}{\eta} - \left( 1 - \frac{1}{\kappa} \right) \eta \right) \right) \\
+ p \gamma (c - H) e^{-r} \left( N \left( \frac{\ln \frac{\nu (H-c)^-}{\lambda \gamma} + (r + \frac{1}{2} \eta^2)}{\eta} - \left( 1 - \frac{1}{\kappa} \right) \eta \right) - N \left( \frac{\ln \frac{\nu (H)^-}{\lambda \gamma} + (r + \frac{1}{2} \eta^2)}{\eta} - \left( 1 - \frac{1}{\kappa} \right) \eta \right) \right) \\
\right) \\
\tag{25}
\]

where \( N(x) := \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt \). The optimal value for \( \lambda \) can be determined numerically from the above equation.\(^{11}\)

4.3. **Appendix C: Lemmas used in the proof of Theorem 2.6.** In the following lemmas, we will use the standard dominated convergence result that if \( x_k \to x \) a.s. (“pointwise”) and for all \( x_k \), \(|x_k| \leq y\) for some \( y \) with \( E[\gamma] < \infty \), then \( E[x] = \lim_{k \to \infty} E[x_k] \).

In this Appendix, as in section 2 of the text, we use the subscript \( \gamma \) to remind us that various Lagrange multipliers depend on \( \gamma \) (since we are looking at asymptotic results in \( \gamma \).

**Lemma 4.1.** The expected cost of repair \( E[\xi \gamma R] \) under the optimal strategy tends to zero as \( \gamma \to 0 \) and as \( \gamma \uparrow \infty \).

**Proof:** The optimal repair is given by (6). Note that for all \( \gamma \), \( 0 \leq R \leq H \). Also note that since \( H - C > \delta \), we have that \( R = 0 \) when \( I_H(\gamma \lambda \gamma \xi) < \delta \), which is when \( \gamma \lambda \gamma \xi > U'_H(\delta) \).

First consider the case \( \gamma \to 0 \). In this case, \( \xi \gamma R \to_{a.s.} 0 \) because a.s. convergence takes \( \xi \) as given and \( R \) is bounded (as we have noted, \( 0 \leq R \leq H \)). Furthermore, for \( \gamma \leq 1 \)

\(^{11}\)Equation (25) can be obtained from the Black-Scholes formula, and particularly the following three equalities resulting from the lognormal distribution of \( \xi \) are useful for the derivation:

\[
E[1_{\{\xi < x\}}] = N \left( \frac{\ln x + (r + \frac{1}{2} \eta^2)}{\eta} \right)
\]

\[
E[\xi 1_{\{\xi < x\}}] = e^{-r} N \left( \frac{\ln x + (r + \frac{1}{2} \eta^2)}{\eta} \right)
\]

\[
E[\xi^a 1_{\{\xi < x\}}] = e^{-a(x + \frac{1}{2} \eta^2 + \frac{1}{2} a^2 \eta^2)} N \left( \frac{\ln x + (r + \frac{1}{2} \eta^2)}{\eta} - a \eta \right)
\]

where \( a \) and \( x \) are constants.
(i.e. far enough out on a sequence with $\gamma \to 0$), $|\xi \gamma R| = \xi \gamma R \leq \xi H$ which has finite expected value because $H$ is a constant and $E[\xi]$ is finite by Assumption 2.1. Therefore, by dominated convergence, $E[\xi \gamma R] = 0$.

Next, consider the case $\gamma \uparrow \infty$. We start with a preliminary result bounding $\lambda_\gamma$. For all $\gamma$, $\lambda_\gamma$ solves the budget constraint with the form of the optimal strategy substituted in:

$$E[\xi I_W(\lambda_\gamma \xi)] + E[\xi \gamma I_H(\gamma \lambda_\gamma \xi)] = W_0.$$ 

For $\gamma = 0$, the second term is zero, and for other (positive) $\gamma$, the second term is positive. The first term is $q(\lambda_\gamma)$, where $q(\cdot)$ was defined in Assumption 2.2 and is decreasing in $\lambda$. Therefore, $\lambda_\gamma \geq \lambda_0 > 0$.

Returning to the case $\gamma \to \infty$, we want to rewrite $E[\xi \gamma R]$ as $(1/\lambda_\gamma)E[\xi \lambda_\gamma \gamma R]$. Since $\lambda_\gamma \geq \lambda_0 > 0$, $(1/\lambda_\gamma) \leq (1/\lambda_0)$ and therefore it suffices to show that $\lim_{\gamma \to \infty} E[\xi \lambda_\gamma \gamma R] = 0$. Observe that $\xi \lambda_\gamma \gamma R \to_{\gamma \to \infty} 0$ a.s. Since, for each $\xi$, $\lambda_\gamma \geq \lambda_0 > 0$ implies that $(\gamma \uparrow \infty) \Rightarrow (\xi \lambda_\gamma \gamma \uparrow \infty)$ and therefore $R = 0$ for sufficiently large $\gamma$ (since $R = 0$ whenever $\gamma \lambda_\gamma \xi > U'_H(\delta)$). Now, $0 \leq R \leq H$ and $R = 0$ whenever $\xi \lambda_\gamma \gamma > U'_H(\delta)$. Therefore, $\xi \lambda_\gamma \gamma R \leq U'_H(\delta)H$, (since for $\xi \lambda_\gamma \gamma > U'_H(\delta)$, $R = 0$, and for $\xi \lambda_\gamma \leq U'_H(\delta)$ we have $0 \leq R \leq H$). Therefore, by dominated convergence, $\lim_{\gamma \to \infty} E[\xi \lambda_\gamma \gamma R] = 0$ and we are done.

**Lemma 4.2.** For the optimal insurance strategy,

$$E[\xi I_W(\lambda_\gamma \xi)] + E[\xi \gamma I_H(\gamma \lambda_\gamma \xi)] = W_0.$$ 

$$E[U_H(H - C - R)] \xrightarrow{\gamma \to 0} U_H(H)$$ 

$$E[U_H(H - C - R)] \xrightarrow{\gamma \to \infty} U_H(H - C).$$

**Proof:** Given $\gamma$, the expected utility of the optimal contract is given by

$$E[U_W(I_W(\lambda_\gamma \xi))] + E[U_H(\min(H, \max(I_H(\gamma \lambda_\gamma \xi), H - C)))].$$ 

For $\gamma$ goes to zero, the argument of $U_H$ converges point-wise to $H$ because $\lambda_\gamma \to \lambda_0$ for $\gamma \to 0$ (see Lemma 4.1), and $I_H(z) \to \infty$ as $z \downarrow 0$. Furthermore, the argument of $U_H$ lies between $\delta > 0$ (by Assumption 2.4) and $H$, so that the absolute value of the expression for $U_H$ is uniformly bounded by $\max(|U_H(\delta)|, |U_H(H)|)$. Therefore, this term converges to $U_H(H)$ and by dominated convergence the expected utility of optimal strategy converges
to $E[U_W(I_W(\lambda_0\xi))] + U_H(H)$ as $\gamma \to 0$.

For $\gamma \to \infty$, the second term of (28) converges to $E[U_H(H - C)]$ since the integrand is bounded and $R_\gamma$ converges point-wise to 0, as shown in the proof of Lemma 4.1.
REFERENCES


