Consensus in Diverse Corporate Boards*

Nina Baranchuk  
University of Texas–Dallas  
Philip H. Dybvig  
Washington University in Saint Louis

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Abstract

A spatial model of preferences over actions illuminates decision-making by boards of directors. Directors have diverse preferences characterized by circular indifference curves around their ideal actions. Board decisions are given by a new model of consensus, which is motivated by the idea that the outcome must be consistent with recourse to majority voting for resolving any disagreements. We derive a number of properties of this model; perhaps most notably we develop the idea that the information a new director brings to a large board is probably more important than the new director’s impact on voting, especially when existing preferences on the board are not too diverse. This implies, for example, that although an executive from a supplier might have preferences far from value maximization for this firm, the executive may still be a valuable director because of the information the executive brings to the board.

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1 Introduction

The corporate board is a cooperative enterprise, since many decisions are made through discussion rather than formal voting. We model board decision-making using a model of consensus that balances weighted local preferences of the directors. We apply the model of consensus to a spatial model of director preferences with concentric indifference curves. This allows us to address questions such as the value of director independence and the impact of strong penalties like those imposed by Sarbanes-Oxley. The analysis illustrates that choosing a strictly independent director may not be optimal, since the information a new director brings to a large board is probably more important than the new director’s vote, especially if the new director’s conflict is not the same as insiders’ conflict. The analysis also shows that strict penalties for deviation from measured value maximization may impair decision-making, especially if the penalty is based on an imperfect measure and inside directors have some commonality of interest with shareholders.

The Nash bargaining solution is the most popular cooperative solution concept. However, the Nash bargaining solution seems inappropriate for a corporate board, since it is predicated on the notion that each party to the negotiation has veto power and can force the outcome to a disagreement point. However, on a board, the threat of a vote limits the influence of individual directors whose preferences are much different from the preferences of the rest of the board. For example, a majority with identical preferences and control of the agenda should be able to dictate the outcome regardless of the preferences of the other directors.

Given that the Nash bargaining solution does not seem appropriate for boards, we introduce a model of consensus. Formally, the consensus sets to zero a weighted sum of the directors’ normalized marginal utility vectors. Intuitively, the consensus balances the local directions (marginal utility vectors) in which agents would like to move the outcomes, without regard to intensity of preference (due to the normalization), understanding that different agents may have different bargaining power (weights). A higher weight may proxy for institutional features we are not modelling explicitly, for example, authority to set the agenda or simply being more persuasive.
Possible board decisions (on all issues) are represented using a spatial model with utility depending on the distance from an ideal point as used in the political model of Baron (1991) and models of product differentiation such as Hotelling (1929). A director’s utility of an action is negative expected squared distance from the director’s possibly random ideal action. These preferences have indifference surfaces that are concentric and the consensus is the weighted sum of distances from the directors’ targets. Because distance is a monotone decreasing function of utility, the weighted sum of distances is a social welfare function for the directors, and consequently the consensus is always Pareto optimal (among the directors, who are not permitted to make side payments). If there is a majority (by weights) of directors with the same target, the consensus is that target, which is sensible because such a set of directors should be able to vote in their preferred action. If all directors’ targets are on a line in action space, then there is a natural ordering of the directors, and the consensus is the target of the weighted median director, a result similar to the median voter theorem.

The consensus also handles reasonably a director with extreme preferences, meaning a director with a target far from the cluster of all the other directors’ targets. Such a director cannot enforce a large deviation from what the other directors want but does have some influence. This is consistent with vote-trading and Baron’s (1991) argument that even a small political party with extreme preferences can obtain some influence by trading votes. Specifically, in this model, an extreme director has an influence that depends on the direction of the target from the consensus, but so long as the target is far from the consensus, not on the distance from the consensus. Intuitively, such a director will spend all available political influence on moving in the direction of the target, which has the same effect whether the target is nearer or farther away.

The model can be used to address a number of policy questions. For example, there has been a lot of attention recently to populating a board with a majority of independent directors. However, a conflicted director may be better than an strictly independent director if the conflicted director also brings information to the board. For example, an officer of a major supplier may bring useful information that is more important than the marginal impact on voting. There is also a question of whether a simple majority of outsiders is good enough. If insiders can dictate the agenda or more gen-
erally have higher bargaining weights than the outsiders, they may be able to obtain their preferred outcome even if in the minority in numbers. And, a minority with concentrated preferences may prevail over a majority with conflicting interests, since the directors in the majority might spend much of their persuasive efforts fighting each other.

Another policy question relates to the stiffening of penalties for deviations from measured value-maximization as seems to be the intent of parts of Sarbanes-Oxley. The ability to improve matters using such a penalty depends a lot on being able to measure board performance \textit{ex post}. If only part of performance can be measured, directors facing the penalty will be driven to optimize the measurable part and neglect the rest as in Holmstrom and Milgrom (1991).

This paper is complementary to existing theoretical work on boards. We have excluded features such as costly effort, delegation, and incentives for sharing information, all of which are studied in the interesting paper by Harris and Raviv (2004).\footnote{We could interpret the actions in our model mechanically to be the choices in an information model: action 1 is the amount of effort spent on information-gathering, action 2 is the action taken contingent on one signal value, action 3 is the level of monitoring, etc. However, we do not have any confidence that the preferences we assume are reasonable given that interpretation.} Another interesting theoretical paper is Hermalin and Weisbach (1998), which looks at CEO retention, the effect of negotiations between the CEO and the board on CEO salary and percentage of insiders on the board,\footnote{Recent regulatory changes are intended to give CEO’s less influence on the selection of directors but it is an open empirical question how effective these changes will be. If the changes are effective, the model of Hermalin and Weisbach may be less relevant in future than it was in the past.} and the incentive for costly information-gathering by directors. In both papers, director preferences are collinear: Harris and Raviv have two types, and Hermalin and Weisbach have preferences determined by one parameter. We do not model the adverse selection and moral hazard features that are the focus of the other two papers, but we do allow a rich diversity of director preferences that allow us to answer questions not addressed in the other papers.\footnote{Our paper seems less related to Cai (2005) and the theoretical part of Aggarwal and Nanda (2004). These papers focus on costly effort, by a team of directors collecting information in the case of Cai (2005) and by the manager who is contracted separately by} Our analysis is also significantly different.
from the analysis of Gomes and Novaes (2005), who compare monitoring by a large shareholder with adding the large shareholder to the board. We do have one theme in common with their paper, which is that the usefulness of adding a conflicted director depends on whether the new director’s conflict reinforces or neutralizes the conflict of existing directors.

It might be asked whether consensus induces directors to conceal their true preferences if they are not already known by the other directors. Unfortunately, the answer is yes in general, not only for our definition of consensus but for any possible decision rule satisfying simple and reasonable conditions (symmetry and invariance under linear transformations that rotate, translate, rescale or flip), as is proven in an impossibility theorem that has the flavor of Arrow’s Impossibility Theorem.

Section 2 presents the model of board decision-making using consensus. Section 3 considers some examples that address policy questions such as the trade-off between independence and information and the desirability of imposing stronger penalties, such as those in Sarbanes-Oxley, for not acting to maximize firm value. Section 4 shows a number of properties of consensus, and serves to justify this decision rule. Section 4 also provides tools that are used to construct the examples in Section 3. Section 5 gives the example showing that directors would like to misrepresent preferences to manipulate the consensus, and this section also provides a general impossibility result showing that no other decision rule satisfies simple symmetry and invariance properties. Section 6 closes the paper.

2 Board Decision-Making

The board’s task is to choose an action \( a \in \mathbb{R}^M \). Director \( n \) seeks to maximize a concave and differentiable expected utility conditional on information \( I_n: u(a) = E[U(a)|I_n] \). In this paper, we do not take a view on how the information is shared among directors, but instead focus on the consequences of such information sharing. Another relatively unrelated paper is Song and Thakor (forthcoming), which analyses how career concerns affect information sharing between the CEO and the board and the CEO’s preferences over board composition.
mation set $I_n$ is developed. This interesting question is addressed in Harris and Raviv (2004). In all our examples, all directors have an incentive to share their information once the board is formed, and are assumed to do so. Given a collection of $N$ directors, we need to have a model of what action the board will select. We let directors have different bargaining power weights $b_n$ that describe their influence on the board’s decision. For example, directors who control the agenda, represent large investors or have better negotiation skills may have more influence on board decisions. Here is our definition of consensus, which we use to model the directors’ decision.

**Definition 1.** Action $a$ is a consensus if and only if there exist $z_n$ such that

\begin{equation}
\sum_{n=1}^{N} b_n z_n = 0,
\end{equation}

where each $z_n \in e_n(a)$, defined by

\begin{equation}
e_n(a) \equiv \begin{cases} 
\{u'_n(a)/\|u'_n(a)\|\} & \text{if } u'_n(a) \neq 0 \\
\{\varepsilon/\|\varepsilon\| \leq 1\} & \text{otherwise}
\end{cases},
\end{equation}

for all $n$.

Intuitively, each director $n$ pulls in the direction of the vector of marginal utilities (the direction of steepest increase of the utility function) with an intensity of $b_n$. If the vector of marginal utilities is zero, the director is at a bliss point and will pull with an intensity of up to $b_n$ in whatever direction is needed to stay at the bliss point. Notice that the definition implies that doubling the bargaining power of an agent has the same impact on the consensus as adding another director with the same preferences. In most of our examples we assume that all directors have the same bargaining power $b_n \equiv 1$.

Consensus can be extended to more general preference orderings as shown in a footnote. However, in this paper we want to specialize further, and

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4Let $P_n$’s be preference orderings for agent $n$ that are irreflexive (for all $x$ and $n$, not $xP_n x$) with convex preferred sets (for all $x$ and $n$, $\{y/yP_n x\}$ is convex), and lower-hemicontinuous. Then we can define consensus action $a^*$ by $\sum_n b_n z_n = 0$ for some $z_n$’s
we assume that each director’s utility is negative expected squared deviation from the director’s possibly random ideal action (ideal) $y_n$:

$(3) \quad u_n(a) = -E[(y_n - a)'(y_n - a)|I_n].$

Conditioning on the director’s information, this utility can be decomposed into two parts: the error in forecasting $y_n$ and the distance to the conditional target

$(4) \quad t_n = E[y_n|I_n]$

from the action $a$ chosen by the board. Specifically,

$(5) \quad u_n(a) = -E[((y_n - E[y_n|I_n]) + (E[y_n|I] - a))'(y_n - E[y_n|I_n])]$

$\quad + (E[y_n|I_n] - a)|I_n]$

$\quad = -E[((y_n - E[y_n|I_n])'(y_n - E[y_n|I_n])|I]

$\quad - 2E[((y_n - E[y_n|I_n])'(E[y_n|I_n] - a)|I_n]

$\quad - E[(E[y_n|I_n] - a)'(E[y_n|I_n] - a)|I_n]

$\quad = -E[((y_n - E[y_n|I_n])'(y_n - E[y_n|I_n])|I_n]

$\quad - (t_n - a)'(t_n - a).$

In the final expression, the first term (error in forecasting $y_n$) does not depend on the action $a$, and the cross term (with the factor 2) was dropped since $E[y_n - E[y_n|I]|I] = 0$ and $E[y_n|I] - a$ is constant given $I$. Therefore, given directors’ information, preferences over actions depend only on the final term $-(t_n - a)'(t_n - a)$.

With utilities given by (3), if the action space $\mathbb{R}^M$ is the plane $\mathbb{R}^2$, a physical interpretation may be useful. Think of the action space as the frictionless surface of a table, with weight $b_n$ for director $n$ suspended by a weightless wire at the director’s target. The wires extend around pulleys to a common

$$z_n \in \text{HULL}\{z||z|| = 1 \text{ and for } yP_n a^*, \ z'(y - a^*) > 0\}. $$
knot on the surface of the table; the equilibrium location of the physical knot is the consensus of our model.

The following theorem offers a useful alternative characterization of the consensus when directors’ preferences minimize expected square deviations from targets.

**Theorem 1. (characterization)** Suppose that director’s preferences are given by the expected square distance loss function (3) with fixed $I_n$ and therefore the target (4). Then, an action $a^*$ is a consensus (according to Definition 1) if and only if the action solves

$$
\min_a \sum_{n=1}^{N} b_n \| t_n - a \|
$$

*Proof.* Given preferences (3), Definition 1 of consensus is the first-order condition for the convex but not-everywhere-differentiable minimization (6). The detailed proof is in the Appendix.

Our definition of consensus is a simple reduced-form that is intended to generate qualitative features that might be found in a more elaborate model with such complex features as agenda-setting and vote-trading. Section 4 quantifies and proves, given utilities (3) or in some cases for general concave preferences, that the consensus in Definition 1 has the following properties:

1. Consensus always exists.
2. Consensus is unique unless the targets are collinear.
3. Consensus is Pareto optimal.\(^5\)
4. When targets are collinear, consensus is the weighted median voter.
5. If there is a majority by weights with the same preferences, the consensus is the majority’s target.
6. A director with extreme preferences has a limited influence on the consensus.

\(^5\)Given Pareto optimality of consensus, it may be natural to ask what is the relationship between the consensus and the core. When the board consists of only two directors, the set of consensus actions coincides with the core. In general, however, it is difficult to compare the two notions, because it is not clear how to define a winning coalition in our setting.
These properties are used in some of the results developed in the following sections. The last property provides an important motivation for our definition of consensus, as it makes the concept particularly applicable to modelling board negotiations. We have in mind the intuition that even a single director with extreme preferences will have some influence because a small concession on an issue this director cares about can be obtained by trading votes with more influential directors, in line with Baron’s (1991) argument why a small political party can have large influence in a parliamentary system dominated by two large parties.

3 Board Composition and Action Choice

3.1 Directors with Extreme Preferences

Our first example illustrates that a director with extreme preferences cannot accommodate those preferences to large degree (since being a large distance away does not increase intensity of the pull $b_n$), but may obtain some influence in exchange for cooperation on issues of importance to the other players. Besides the direct effect on the consensus, a director may also affect decision-making through the information brought to the process. The following example also illustrates the tradeoff between the additional information brought to the board by a new director and a distortion of the consensus action caused by the new director’s bargaining power.

Example 1. (Information vs. Private Interests) Consider a board with $N = N_0 > 2$ directors with ideals located on a circle with radius $r > 0$ in $\mathbb{R}^3$:

$$y_n = (\lambda + \varepsilon)\bar{y} + (0, \sin(2\pi n/N_0), \cos(2\pi n/N_0))r;$$

where $\bar{y}_1 = 0$, the dispersion $r > 0$ is a constant, and the random variables $\lambda \sim \mathcal{N}(1, \sigma^2_\lambda)$ and $\varepsilon \sim \mathcal{N}(0, \sigma^2_\varepsilon)$ are drawn independently. We assume that these initial $N_0$ directors have no knowledge of $\lambda$ or $\varepsilon$ and have the same bargaining power $b_n = 1$. 

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Suppose another director $N_0 + 1$ with bargaining power $b_{N_0+1}$ joins the board. We assume this director knows $\lambda$ but has no knowledge of $\varepsilon$, and the new director’s preferences diverge, perhaps significantly, from the preferences of the other directors. The new director is assumed to have the ideal

$$y_{N_0+1} = \bar{y}(\lambda + \varepsilon) + (h, 0, 0),$$

where the idiosyncracy $h > 0$. The new director is assumed to share the knowledge of $\lambda$ with the other directors (indeed in this example the director will want to share the information).

Because the initial set of $N_0$ directors has no knowledge of $\lambda$ and $\varepsilon$, by (4) director $n$’s target is

$$t_n = E[y_n] = \bar{y}E[\lambda + \varepsilon] + (0, \sin(2\pi n/N_0), \cos(2\pi n/N_0))r = \bar{y} + (0, \sin(2\pi n/N_0), \cos(2\pi n/N_0))r$$

After the new director joins the board and shares the information about $\lambda$, each original director ($n \leq N_0$) will have the target

$$t_n = E[y_n | \lambda] = \bar{y}\lambda + (0, \sin(2\pi n/N_0), \cos(2\pi n/N_0))r,$$

and the new director’s target is

$$t_{N_0+1} = E[y_{N_0+1} | \lambda] = \bar{y}\lambda + (h, 0, 0).$$

Lemma 1. (Extreme Director) In Example 1, for the initial set of $N_0$ directors, the consensus is located at the center of the circle of their targets:

$$a^*_0 = \bar{y}.$$
The majority rule result (Theorem 4 in Section 4) implies that if \( b_{N_0 + 1} \geq N_0 \), then the board including the new director has a consensus at the new director’s target: \( a^* = t_{N_0 + 1} \). If, on the other hand, \( b_{N_0 + 1} < N_0 \), the consensus is

\[ a^* = \bar{y} \lambda + (h_0, 0, 0), \]

where

\[ h_0 = \min \left( h, \frac{rb_{N_0 + 1}}{\sqrt{N_0^2 - b_{N_0 + 1}^2}} \right). \]

Proof. See the Appendix. \( \square \)

Lemma 1 implies that, in the absence of new information (\( \sigma_\lambda = 0 \) so that \( \lambda \equiv 1 \)), the new director moves the consensus by \( h_0 \). It is perhaps obvious that \( h_0 \) should be small when the new director’s idiosyncracy \( h \) and bargaining power \( b_{N_0 + 1} \) are small. More interestingly, \( h_0 \) is also small when there are many original directors with small dispersion. Theorem 6 in Section 4 generalizes this result.

Lemma 2. (Information vs. Private Interests) In Example 1, after the new director joins, the original directors’ utilities change by

\[ u_n(a^*) - u_n(a_0^*) = \|\bar{y}\|^2 \sigma_\lambda^2 - h_0^2. \]

Proof. From (3), the utility of an original director \( n \leq N_0 \) in the initial consensus is

\[ u_n(a_0^*) = -\|\bar{y}\|^2 (\sigma_\lambda^2 + \sigma_\varepsilon^2) - r^2. \]

and the utility of the original director \( n \) after the new director \( N_0 + 1 \) joins is

\[ u_n(a^*) = -\|\bar{y}\|^2 \sigma_\varepsilon^2 - (r^2 + h_0^2). \]

\( \square \)
Lemma 2 implies that the original directors’ utilities may be higher or lower after the new director joins, depending on whether $\|\tilde{y}\|^{2}\sigma_{\lambda}^{2}$ is larger or smaller than $h_{0}^{2}$. We can interpret $\|\tilde{y}\|^{2}\sigma_{\lambda}^{2}$ as the value of the new information and we can interpret $h_{0}^{2}$ as the cost of the distortion from the new director’s bargaining power. The value of the new information $\|\tilde{y}\|^{2}\sigma_{\lambda}^{2}$ is the contribution of $\lambda$ to the variance of each original director’s ideal. The cost of the distortion $h_{0}^{2}$ is the squared distance from the new consensus $a^{*}$ to the action that would be chosen by the original directors if they knew $\lambda$ but did not have to let the new director participate in decision-making.

3.2 Director Independence

Conflicts of interest may arise when directors have ties (beyond board membership) to managers they are meant to supervise. This seems to imply that director independence is crucial for board effectiveness and suggests that regulations encouraging board independence may be beneficial. In this subsection, we use our model to explore issues of director independence and we consider circumstances in which the current definition of independence would be too strict.

In the United States, regulation of the composition of corporate boards has been largely delegated to Self-Regulatory Organizations (SROs) such as the NYSE and NASD. Both of these organizations have recently adopted rules requiring a majority of the board to be composed of independent directors. According to NASD rule 4350(c),

A majority of the board of directors must be comprised of independent directors as defined in Rule 4200.

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6 Self-regulation of industry, operating under the threat that Congress and governmental agencies will take over with more severe and less efficient regulation, is intended to be an effective, efficient, and cost-effective alternative to direct regulation by governmental agencies such as the SEC.

7 The NASD rules are quoted from the “Rules of the Association” in the NASD Manual, which is available online at http://www.nasd.com. The quotes are current as of July 11, 2005.
where the definition of independence excludes employees, high-paid consultants, close relatives of executives, and significantly for our purposes executives of a significant customer or supplier. Specifically, NASD rule 4200-1(a)(14)(C) says a person satisfying the following description should not be considered independent:

a director who is a partner in, or a controlling shareholder or an executive officer of, any for-profit business organization to which the corporation made, or from which the corporation received, payments (other than those arising solely from investments in the corporation’s securities) that exceed 5% of the corporation’s or business organization’s consolidated gross revenues for that year, or $200,000, whichever is more, in any of the past three years.

The NYSE has a similar rule. According to NYSE rule 303A.01,8

Listed companies must have a majority of independent directors,

and according to NYSE rule 303A.02(b)(v), the director is not considered independent if

the director is a current employee, or an immediate family member is a current executive officer, of a company that has made payments to, or received payments from, the listed company for property or services in an amount which, in any of the last three fiscal years, exceeds the greater of $1 million, or 2% of such other company’s consolidated gross revenues.

Our model can be used to show circumstances in which these definitions of independence are too strict and can rule out a director who would be a useful addition to the board.

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8The NYSE rules are quoted from the NYSE Listed Company Manual, which is available online at http://www.nyse.com. The quotes are current as of July 11, 2005.
Example 2. There is a vacancy on a board, and we want to consider the choice between two candidates. The first candidate $C_1$ is a strictly independent director who wants to maximize firm value but brings no information. The second candidate $C_2$ is somewhat conflicted but also brings some information to the board, for example, the second candidate may be part owner and an executive of a major supplier.

The action space is $\mathbb{R}^M = \mathbb{R}^5$; having five separate actions allows us to assume separability of preferences in the example so we can look at cross effects in the solution that are present even in the absence of any direct effects in preferences. Besides the vacancy, there are $N_o \geq 2$ slots on the board held by existing outsiders and $N_i > 0$ slots held by existing insiders. All directors have the same bargaining power $b_n = 1$. Existing outside directors $n = 1, \ldots, N_o$ have ideals

\begin{equation}
y_n = (0, 0, r \sin(2\pi n/N_o), r \cos(2\pi n/N_o), \lambda + \varepsilon),
\end{equation}

where the random variables $\lambda \sim \mathcal{N}(1, \sigma_\lambda^2)$ and $\varepsilon \sim \mathcal{N}(0, \sigma_\varepsilon^2)$ are drawn independently, and the sine and cosine functions have been chosen to space the existing outsiders evenly along a circle with radius $r > 0$ centered at the firm value maximizing action $(0, 0, 0, 0, \lambda + \varepsilon)$. The last coordinate $\lambda + \varepsilon$ is common to all agents; $\lambda$ is a random variable known by candidate $C_2$ but none of the other directors, while $\varepsilon$ is an error term not known to anyone.

Existing inside directors, $n = N_o + 1, \ldots, N_o + N_i$, all fear retribution from the CEO and act exactly as the CEO would like, with identical ideals

\begin{equation}
y_n = (\delta, 0, 0, 0, \lambda + \varepsilon),
\end{equation}

for some $\delta > 0$.

The first candidate new director $C_1$ is strictly independent which we interpret as wanting to maximize profits. Therefore, $C_1$ has ideal

\begin{equation}
y_{N_o+N_i+1}^1 = (0, 0, 0, 0, \lambda + \varepsilon).
\end{equation}

The second candidate director $C_2$ is somewhat conflicted but also brings knowledge of $\lambda$ to the board. The candidate $C_2$ has ideal

\begin{equation}
y_{N_o+N_i+1}^2 = (g_1, g_2, 0, 0, \lambda + \varepsilon),
\end{equation}
with both \( g_1 \neq 0 \) and \( g_2 \neq 0 \). Both candidates have the same bargaining power \( b_n = 1 \). We assume that the objective of maximizing firm value is given by (3) with the ideal \((0, 0, 0, 0, \lambda + \varepsilon)\).

In this example, because insiders collude (have the same ideal) while independent directors have dispersed ideals, the consensus action may coincide with the insiders’ target even if there is a majority by weights of independent directors. Notice that the third and fourth consensus actions are zero with or without any new candidate. Calculations similar to those in Lemma 1 imply that if the majority of the board is independent: if \( N_i < N_o \), then the consensus action without any new candidate is

\[
a^*_0 = (h_0, 0, 0, 0),
\]

where \( h_0 = \min(\delta, rN_i/\sqrt{N_0^2 - N_i^2}) \), which coincides with the insiders’ target when \( h_0 = \delta \).

**Lemma 3. (Ineffective Majority of Independent Directors)** Adding any new director, including the strictly independent candidate \( C_1 \), preserves the consensus at the insiders’ target if

\[
(N_i^2 - 1)\sqrt{1 + r^2/\delta^2} - \frac{N_o}{2\sqrt{1 + r^2/\delta^2}} \geq 1.
\]

*Proof.* See the Appendix.

For any given board composition \( N_o \) and \( N_i \), the above inequality is satisfied when the disagreement among independent directors \( r \) is large enough compared to their disagreement with the insiders \( \delta \).

The intuition from Example 1 suggests that adding the conflicted but informed candidate \( C_2 \) results in a higher firm value than adding a strictly independent candidate \( C_1 \) if the reduction in the uncertainty from the new information compensates for the bias in the action choice. Because it is difficult to find a closed-form expression for the consensus with the new candidate \( C_2 \), we illustrate this intuition using numerical solution with the following parameter values: \( N_o = 4, N_i = 3, \delta = 1, \) and \( r = 0.3 \).
Figure 1: **Illustration for Example 2** A candidate informed director whose conflict is closely aligned with insiders (as it is on the right in this figure) requires a larger amount of information $\sigma_\lambda^2$ to be a good addition to the board than a director who is unconflicted or has a conflict opposing the insiders (on the left).

For any new director of type $C_2$, the third and the forth coordinates of the consensus action are zero and the fifth coordinate is $\lambda$. Figure 1 shows the projection of our solution on the first two coordinates. The projection of all possible consensus actions with the new candidate is the area bounded by the dotted line. Each solid line shows the locus of the new director’s targets that result in the consensus actions with the same distance from the value maximizing action $(0, 0, 0, 0, \lambda + \varepsilon)$. If the new director’s target is located in the cone formed by rays $r_1$ and $r_2$, then the consensus coincides with the insider’s target.
Utility decomposition (3) implies that, for each locus, there is a threshold value of $\sigma^2_3$ (indicated on the graph), above which adding the new informed director with a target on the locus results in a higher firm value than adding the strictly independent candidate $C_1$. The figure shows that the firm owners may prefer to add a director with a significant conflict of interest (large $\sqrt{g^2_1 + g^2_2}$) and little new information (small $\sigma^2_3$) if the director’s conflict of interest differs from that of the existing insiders ($g_2$ may be large, but $g_1$ is positive and small or negative).

We have seen that insiders acting on behalf of the CEO may get what they want in spite of a majority of outsiders. In the model, that can happen because the outsiders have diverse interests centered around but far from the optimum (when $r$ is larger) or when insiders have larger bargaining weights (perhaps as a proxy of agenda-setting power). In practice, a minority of insiders could maintain control for other reasons not in the model. For example, insiders seem likely to have better attendance at meetings and could have a majority of a quorum at most meetings but a minority of the whole board. Or, an outside director may be a CEO or ex-CEO who thinks of board meddling as intrusive and counterproductive. Such an outsider would side with management and would also tend to select like-minded candidates when serving on a nominating committee.

### 3.3 Fiduciary Duties

Incentives or penalties for deviations from the value-maximizing actions can be used to mitigate the conflicts of interests that influence board’s actions. Stronger incentives seem desirable when directors face more conflicts of interests. The current legal system imposes fiduciary duties (duty of diligence, loyalty, and obedience) on directors. While directors are generally protected by the business judgment rule, stricter standards are applied when apparent conflicts of interest are present. According to Knepper and Bailey (2004),

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9The business judgment rule, as stated by the Delaware courts, is “a presumption that in making a business decision the directors of a corporation acted on an informed basis, in good faith and in the honest belief that the action taken was in the best interests of the company.”
When a challenger of the business judgment rule has shown that directors have a self-interest in the transaction at issue – that is, that the corporate fiduciaries, because of a conflict are disabled from safeguarding the interests of the shareholders to whom they owe a duty – the burden of proof shifts to the directors.

For similar reasons, inside directors may also be subject to more scrutiny:

Under numerous statutory theories of liability, an “inside” director faces greater liability exposure than a disinterested “outside” director.

The extent to which the courts impose fiduciary duties is limited; courts generally do not “second-guess the directors’ choice of procedure absent gross negligence” (Knepper and Bailey (2004)). In the recent years, however, the penalties for corporate wrongdoing have increased, primarily due to Sarbanes-Oxley Act of 2002.\(^\text{10}\) Sarbanes-Oxley also prohibits certain actions that are deemed to create conflicts of interest such as loans to directors and officers.

In our model, the implications of penalizing deviations from the value maximizing action can be illustrated as follows. Suppose that directors’ preferences have two parts: deviation from the ideal action \(y_n\) and a penalty for deviation from some possibly random action \(a_0\),

\[
(15) \quad u_n = -E[(y_n - a)'(y_n - a) - \beta_n(a_{0n} - a)'(a_{0n} - a)|I_n],
\]

where \(\beta_n > -1\), or else there is not a unique \(a\) that maximizes (15). The penalty \(\beta_n(a_{0n} - a)'(a_{0n} - a)\) can arise from various sources such as compensation, requirements imposed by SROs, governmental regulation and law enforcement. Action \(a_0\) may depend only on the information available to the party that imposes the penalty. Because the penalty is applied after the board decision-making is over, \(a_{0n}\) may in particular depend on some information unavailable to directors during the decision making process. In that

\(^{10}\)According to Bloomenthal (2003), while Sarbanes-Oxley did not change the existing scheme of penalties substantially, it has probably increased the likelihood of going to prison for violations of securities laws.
case, director \( n \) views \( a_{0n} \) as random when determining the target action. From (15),

\[
(16) \quad u_n = -E[(y_n - a)'(y_n - a)|I_n] - \beta_n E[(a_{0n} - a)'(a_{0n} - a)|I_n]
\]

\[
= -E[(y_n - a)'(y_n - a)|I_n] - \beta_n (a'a - 2a' E[a_{0n}|I_n] + E[a_{0n}'a_{0n}|I_n])
\]

\[
= -E[(y_n - E[y_n|I_n])'(y_n - E[y_n|I_n])|I_n] - \beta_n E[(a_{0n} - E[a_{0n}|I_n])'(a_{0n} - E[a_{0n}|I_n])|I_n]
\]

\[
- \frac{\beta_n}{1 + \beta_n} (E[y_n|I_n] + E[a_{0n}|I_n])'(E[y_n|I_n] + E[a_{0n}|I_n])
\]

\[
- (1 + \beta_n) \left( \frac{E[y_n + \beta_na_{0n}|I_n]}{1 + \beta_n} - a \right)' \left( \frac{E[y_n + \beta_na_{0n}|I_n]}{1 + \beta_n} - a \right)
\]

Note that only the last term depends on the chosen action \( a \). Therefore, given preferences with the penalty in (15), the target is

\[
(17) \quad t_n = \frac{E[y_n|I_n] + \beta_n E[a_{0n}|I_n]}{1 + \beta_n}.
\]

and cardinal preferences are an affine transform of what they were in (5) so that ordinal preferences\(^{11}\) are the same as they were in (5). Note that when \( \beta_n > 0 \), (16) implies that the utility of director \( n \) is decreasing in the conditional volatility \( E[(a_{0n} - E[a_{0n}|I_n])'(a_{0n} - E[a_{0n}|I_n])|I_n] \) of the benchmark \( a_{0n} \), given the conditional mean \( E[a_{0n}|I_n] \) and the information structure.

**Lemma 4.** Consider a board where directors 1, 2, ..., \( N_0 \) have the same random target \( t \) in the absence of penalties. Suppose these directors face the same penalty \( \beta(a_0 - a)'(a_0 - a) \) and have more bargaining power than the rest of the board: \( \sum_{n=N_0+1}^{N} b_n > \sum_{n=1}^{N_0} b_n \). Then \( a^* = t_n \) given by (17) for \( n \leq N_0 \). Suppose \( a_0 \) is chosen as an arbitrary finite-variance function of some information set available ex post, and that \( \beta \) is chosen to be greater than \(-1\), with the objective of maximizing firm value or equivalently minimizing \( E[(a^* - a_{fv})^2] \). Then, \( \beta \) and \( a_0 \) are optimal if and only if they shift the expected target of directors \( n \leq N_0 \) to \( E[a_{fv}] \):

\[
(18) \quad E[a_{fv}] = \frac{E[t] + \beta E[a_0]}{1 + \beta}
\]

\(^{11}\)Ordinal preferences concern the ordering of outcomes, in contrast to cardinal preferences that concern the actual utility numbers. A monotone transformation of the utility function changes cardinal preferences but not ordinal preferences.
and minimize the variance

\begin{equation}
V \left( a_{fv} - \frac{t + \beta E[a_0|I_n]}{1 + \beta} \right)
\end{equation}

of the deviation of the majority directors’ target from the full value action.

Proof. Because directors \( n \leq N_0 \) have the majority of the bargaining power and identical targets, the consensus is their common target from (17), i.e. \( a^* = (t + \beta E[a_0|I_n])/(1 + \beta) \). Therefore, the expected squared deviation of \( a^* \) from \( a_{fv} \) can be rewritten as

\[
E[(a_{fv} - a^*)^2] = E[(a_{fv} - E[a_{fv}] + E[a_{fv}] - E[a^*] + E[a^*] - a^*))^2]
\]
\[
= E[(E[a_{fv}] - E[a^*])^2] + E[(a_{fv} - a^* - (E[a_{fv}] - E[a^*]))^2]
\]
\[
= E \left[ \left( E[a_{fv}] - \frac{E[t] + \beta E[a_0]}{1 + \beta} \right)^2 \right] + V \left( a_{fv} - \frac{t + \beta E[a_0|I_n]}{1 + \beta} \right).
\]
\[
= E \left[ \left( E[a_{fv}] - \frac{E[t] + \beta E[a_0]}{1 + \beta} \right)^2 \right] + V \left( a_{fv} - \frac{t + \beta(E[a_0|I_n] - E[a_0])}{1 + \beta} \right).
\]

Consider the last expression. The first term is minimized at 0 when (18) is satisfied and the second term is (19). If \( \beta \) and \( a_0 \) satisfy (18) and minimize (19), both terms are minimized and therefore the sum is minimized. Conversely, we want to show that if \( \beta \) and \( a_0 \) do not satisfy (18) or do not minimize (19), then the sum of both terms is not minimized.

Suppose \( \beta \) and \( a_0 \) do not satisfy (18). If \( \beta \neq 0 \), then adding a constant to \( a_0 \) to satisfy (18) reduces the first term and does not affect the second term, showing that \( \beta \) and \( a_0 \) did not minimize the sum. If \( \beta = 0 \), then changing \( \beta \) to any nonzero value and adding a constant to \( a_0 \) to satisfy (18) reduces the first term to zero. The resulting change in the second term can be made arbitrarily small by setting \( \beta \) to a small enough value. Therefore, small enough \( \beta \) and corresponding \( a_0 \) to satisfy (18) must reduce the overall sum, showing that the original values did not minimize the sum.
If $\beta$ and $a_0$ do not minimize (19), then changing $\beta$ to some nonzero value that reduces the second term and adding a constant to $a_0$ to satisfy (18) reduces the second term and does not increase the first term, showing that $\beta$ and $a_0$ did not minimize the sum.

Notice that, for any $\beta \neq 0$ (which is without loss of generality as noted in the proof), $E[a_0]$ can be set to $(E[a_{fv}](1 + \beta) - E[t]) / \beta$ to satisfy (18). Therefore, the strength of penalty $\beta$ and the random part of the target $E[a_0|I_n] - E[a_0]$ can be chosen to minimize (19). Minimizing the variance (19) is like a linear regression. Given $\beta > -1$, $\beta \neq 0$, the choice of $a_0$ gets rid of the volatility in the $a_{fv}$ and $t$ terms that can be spanned by the information available in $E[a_0|I_n]$ for different choices of $a_0$. The determination of $\beta$ therefore is related to the parts of $a_{fv}$ and $t$ that cannot be spanned. If, for example, the parts of $a_{fv}$ and $t$ that cannot be spanned are negatively correlated, minimization of the variance would want to pick $1/(1 + \beta)$ to be negative, which is infeasible. In this case, there is a closure problem and the optimal value would be approached in the limit as $\beta \uparrow \infty$: since it is infeasible to use $t$ to hedge the risk in $a_{fv}$, the best that can be done is to “kill off” the term in $t$ by making the denominator large.

Lemma 4 assumes that the penalty can be implemented at no cost. In reality, imposing small penalties and fines can be considered costless or even beneficial for the rest of the society, but more serious punishments such as imprisonment are fairly costly. The penalty may also impose an indirect cost on the firm, because it reduces directors’ utility and, in the absence of rents, would lead directors to require higher compensation for their services.

Implementing the penalty suggested in Lemma 4 may be difficult, since it requires an ability to identify what actions would increase the firm value. Indeed, the empirical studies that analyze the link between various board characteristics and firm value typically rely on such obviously flawed measures of board performance as Tobin’s Q, the “universal proxy” (see, for example, Yermack (1996)). As shown in Holmstrom and Milgrom (1991), when there is a lot of uncertainty about what maximizes the firm value, any

\footnote{If moving $\beta$ to 0 reduces the second term, continuity of the second term implies that there is a neighborhood of 0 such that moving $\beta$ anywhere in the neighborhood also reduces the second term, so without loss of generality we can choose a nonzero value.}
incentive scheme is likely to cause undesirable biases in the agents’ choices.

3.4 Information Sharing

In Examples 1 and 2, the new information is beneficial to all directors. This is not always the case. As illustrated by the following example, a majority may use the new information to implement a policy that damages the minority.

Example 3. (Disutility From Information) Consider a board with an odd number \( N = 2N_0 - 1 \) of directors with the same bargaining power \( b_n = 1 \) and ideals in \( \mathbb{R} \) given by

\[
y_n = (\lambda + \varepsilon)\beta_n
\]

for \( n = 1, \ldots, N \), where \( \beta_1 \leq \beta_2 \leq \ldots \leq \beta_N \), and the random variables \( \lambda \sim \mathcal{N}(0, \sigma_\lambda^2) \) and \( \varepsilon \sim \mathcal{N}(0, \sigma_\varepsilon^2) \) are drawn independently.

If the directors have no knowledge of \( \lambda \) or \( \varepsilon \), from (4), all directors have the same target \( t_n = 0 \) and consequently this board chooses the consensus \( a_0^* = 0 \). From (3), director \( n \)'s utility is then

\[
u(a_0^*) = -\left(\sigma_\lambda^2 + \sigma_\varepsilon^2\right)\beta_n^2.
\]

If, on the other hand, the directors know \( \lambda \), the assumed ordering of \( \beta \)'s and medium voter Theorem 5 (see Section 4) imply that the new consensus is \( a^* = \lambda\beta_{N_0} \). In this case, director \( n \)'s utility is

\[
u(a^*) = -\sigma_\varepsilon^2\beta_n^2 - \lambda^2(\beta_n - \beta_{N_0})^2.
\]

Hence, director \( n \)'s utility decreases when the board learns \( \lambda \) if \( \beta_n / \beta_{N_0} < 1/2 \). In other words, the utility decreases for directors whose targets are closer to zero than to the median voter’s. In particular, if \( \beta_n = 0 \), this director is worse off. If the target of \( \alpha = 0 \) is consistent with firm value maximization, then the new information in this example decreases firm value (provided \( \beta_{N_0} \neq 0 \)). In that case, the information only serves directors’ interests at the expense of shareholders.
This example also illustrates that sharing private information with the rest of the board may not be in the director’s interest. Specifically, suppose that the value of \( \lambda \) is known only to the director \( n \). Then, the informed director benefits from informing the board about the value of \( \lambda \) only if \( \beta_n > \beta_{N_0}/2 \).

4 Properties of Consensus

Decision-making on a corporate board is a complex dynamic process that includes such institutions as formal votes, agenda-setting, coalition formation, and vote-trading. Like the Nash bargaining solution, our analysis does not attempt to build a detailed structural model and instead provides a simple and robust mechanism that captures the economics of the situation but is more tractable than a detailed structural model would be. Our model of consensus is more appropriate for a corporate board or other committee than the Nash bargaining solution would be, since the Nash bargaining solution is predicated on the assumption that unanimity is required so that the solution can only improve on a given disagreement point. In a corporate board or other committee with majority voting, the outcome must do well by majorities, even if the outcome is very undesirable for some minority members.

For the application in this paper, we always assume that director’s preferences are given by the expected square distance loss function (3). Given the information sets \( I_n \) and ideals \( y_n \), targets are determined (through the calculation in (5)) and determine the preferences and solution. We derive properties of consensus for this special case, stated in terms of the target, and we note after each result the extent to which it still holds for general concave preferences. An exception is the result for Pareto optimality in Theorem 3, which is stated generally since the proof is almost the same in that case.

First, we show that consensus always exists and the consensus is unique except in a degenerate case. (It is also easy to modify the definition of consensus—using a measurable selection—to be unique all the time, for example, by selecting the midpoint of any interval of solutions.)

Theorem 2. (Existence and Uniqueness) Given any \( t_1, \ldots, t_N \), there
exists a consensus defined in Definition 1. Provided not all the $t_1, \ldots, t_N$ are collinear, consensus is unique.

**Proof.** By Theorem 1, consensus minimizes the objective (6). Consensus exists because the objective in (6) is continuous and dominated outside a compact set. If the $t_n$’s are not collinear, the objective in (6) is strictly convex, which implies uniqueness. The detailed proof is in the Appendix. \[\Box\]

Existence of consensus can be generalized to Definition 1 with some additional structure. Some assumption is required to restrict attention to a compact set; once we do this the upper-hemicontinuity of the derivative correspondence can be used with Kakutani’s theorem to prove existence.\(^{13}\) The uniqueness result does not seem to be general.

**Theorem 3. (Pareto Optimality)** If utilities are concave, differentiable, and have a unique maximum (bliss point) or no maximum, then a consensus defined in Definition 1 is Pareto optimal. In particular, with a target $t_n$, the bliss point is unique and consensus is Pareto optimal.

**Proof.** Let $a^*$ be the consensus. If for some $n$, $u_n'(a^*) = 0$, then $a^*$ is the unique maximum of $u_n$: any $a \neq a^*$, $u_n(a) < u_n(a^*)$. Therefore, $a^*$ is Pareto optimal. If, on the other hand, $u_n'(a^*) \neq 0$ for all $n$, consider the following social welfare function,

$$W(a) = \sum_{n=1}^{N} \frac{b_n}{\|u_n'(a^*)\|} u_n(a).$$

Because utilities $u_n(a)$ are concave, $W(a)$ is also concave. Therefore, actions that satisfy the first-order condition $W'(a) = 0$ are Pareto optimal. According to Definition 1, $a^*$ satisfies (1), which is equivalent to $W'(a^*) = 0$. \[\Box\]

If a group of directors has the same target and more total bargaining power than the rest of the board, then their target is the consensus.

\(^{13}\)Some care has to be taken at the boundary; we want to map an action into the action plus the sum of the gradients to find a fixed point. However, this can map outside our chosen compact set. This can be fixed by extending the compact set and using the projection to the original compact set when we start outside.
Theorem 4. *(Majority Rule)* Suppose directors $n = 1, ..., N_0$ have the same target $t$.

1. If directors $1, ..., N_0$ have more bargaining power than the rest of the board: $\sum_{n=1}^{N_0} b_n > \sum_{n=N_0+1}^{N} b_n$, then $t$ is the unique consensus.
2. If directors $1, ..., N_0$ have the same bargaining power as the rest of the board: $\sum_{n=1}^{N_0} b_n = \sum_{n=N_0+1}^{N} b_n$, then $t$ is a consensus. It is unique if not all the $t_1, ... t_N$ are collinear.

*Proof*. The majority at $t$ can have $z_n$’s offsetting all the other $z_n$’s in the characterization in Theorem 1. The detailed proof is in the Appendix.  \qed

The result on majority rule is general, with almost the same proof, except for uniqueness in the second part.

The following result shows that when targets are collinear, a version of the median voter result holds.

Theorem 5. *(Median Voter)* Suppose all directors have collinear targets: $t_n = a_0 + \rho_n(a_1 - a_0)$, where $\rho_1 \leq \rho_2, ..., \leq \rho_n$. Let $N_0$ be such that

\[
\sum_{n=1}^{N_0-1} b_n \leq \frac{1}{2} \sum_{n} b_n
\]

and

\[
\sum_{n=1}^{N_0} b_n > \frac{1}{2} \sum_{n} b_n.
\]

1. If (20) holds with inequality, then $t_{N_0}$ is the unique consensus.
2. If (20) holds with equality, then $[t_{N_0-1}, t_{N_0}]$ is the set of consensus actions, which is a single point if and only if $t_{N_0-1} = t_{N_0}$.

*Proof*. See the Appendix. \qed

In the theorem, the weighted median voter’s target is not unique if it is possible to strictly separate two groups with equal total weight that prefer
to move in opposite directions. For example, consensus is never unique if there is an even number of directors with equal weights and distinct collinear targets. In these cases, every action in the closed interval between the two groups is a consensus.

The median voter result is general only in the case of a one-dimensional action space. With a higher-dimensional action space and general concave preferences, collinear targets do not imply collinear marginal utility vectors, and the result fails.

The following theorem generalizes the result of Lemma 1, which offers a bound on an extreme director’s impact on the consensus.

**Theorem 6. (Extreme Director)** Consider a board with \( N > 2 \) directors with targets \( t_1, ..., t_N \) and same bargaining power \( b_n = 1 \) for all \( n \). Suppose that for all \( i, j < N - 1 \), \( \|t_i - t_j\| \leq K \) and let \( a^* \) be the consensus. Then,

\[
\text{d}(H, a^*) \leq \frac{K}{N - 2},
\]

where \( \text{d}(H, a^*) \equiv \min_{h \in H} (\|h - a^*\|) \) is the distance from \( a^* \) to a compact set \( H \), \( H = \text{hull}(t_1, ..., t_{N-1}) \), and \( \text{hull} \) denotes the convex hull.\(^{14}\)

**Proof.** See the Appendix. \( \square \)

This result seems to be special to the preferences with concentric indifference curves around targets.

We devote the rest of the section to the result that while directors have utility functions that are separable across actions,

\[
U_n(a) = \|a - t_n\|^2 = (a_1 - t_{n1})^2 + (a_2 - t_{n2})^2 + ... + (a_M - t_{nM})^2,
\]

where \( a = (a_1, ..., a_M) \) and \( t_n = (t_{n1}, ..., t_{nM}) \), consensus is not separable on the actions. This result is illustrated in the following example.

\(^{14}\)Recall that the convex hull of a finite set \( X \) is a set of all convex combinations of elements of \( X \).
Example 4. (Action Separability) Consider a board with 4 directors with no uncertainty about their ideals. Let their ideals (same as targets) be: \( t_1 = (-1, 0) \), \( t_2 = (2, 0) \), \( t_3 = (0, -1) \), and \( t_4 = (1, h) \). All directors have the same bargaining power \( b_n = 1 \). The consensus is given by

\[
\begin{align*}
\alpha^* &= \begin{cases} 
(1/(1+h), 0) & \text{for } h \geq 0; \\
(1, h) & \text{for } 0 \geq h \geq -\frac{1}{2}; \\
\left(\frac{2+h}{1-h}, \frac{3h}{2(1-h)}\right) & \text{for } -\frac{1}{2} \geq h \geq -2; \\
(0, -1) & \text{for } -2 \geq h,
\end{cases}
\end{align*}
\]

as can be verified by checking the first-order conditions given by Theorem 1.

The consensus to the Example 4 is illustrated in Figure 2. Curve \( ABCD \) is
the locus of the consensus actions obtained for different values of $h$ (different locations of $t_4$). When $h$ is positive, the consensus is at the intersection of the line connecting $t_1$ and $t_2$ and the line connecting $t_3$ and $t_4$ (point $X_1$). Similarly, when $-1/2 < h < -2$, the consensus is at the intersection of the line connecting $t_1$ and $t_4$ and the line connecting $t_2$ and $t_3$. When $-1/2 < h < 0$, $t_4$ is located inside the triangle formed by $t_1$, $t_2$, and $t_3$, so the consensus is at $t_4$; for $h < -2$ ($t_4$ below $X_2$), target $t_3$ falls inside the triangle formed by $t_1$, $t_2$, and $t_4$, so the consensus is at $t_3$.

Although directors’ preferences are separable across actions and changing $h$ affects only one director’s preferences for action $a_2$, the consensus action $a_1^*$ changes, and not even in a monotone way. However, this is reasonable, and there is a simple explanation: when either $h \gg 0$ or $h \ll 0$, director 4’s strong preferences for $a_2$ imply that the director focuses political capital on $a_2$. Therefore, director 4’s preference to move $a_1^*$ to the right has little impact when $|h|$ is large and large impact when $|h|$ is small.

For the general case of Definition 1, preferences are not separable; if they are, the solution typically is not just as in Example 4.

5 An Impossibility Theorem

Our model assumes that the targets of each director are publicly known. If the information about these locations were not public, however, truthful reporting would likely not be in the directors’ best interests. Given our model setup, it might be natural for each director $n$ to pretend to have a target $\hat{t}_n$ that minimizes the distance from the consensus to the true target $t_n$. The following example illustrates how directors may benefit from misreporting their targets. Note that, throughout this section we assume that all directors have the same unit bargaining power.\footnote{This is without loss of generality because, as we mentioned in Section 2, one director with bargaining power $b_n$ is equivalent to $b_n$ directors with unit bargaining power.}

Example 5. (Truthful Reporting?) Consider a board with 4 directors

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whose targets are
\[ t_1 = (t_{11}, 0) \]
\[ t_2 = (-\alpha t_{11}, 0) \]
\[ t_3 = (t_{21}, t_{22}) \]
\[ t_4 = (-\beta t_{21}, -\beta t_{22}) \]

where \( \beta > 0 \), \( \alpha > 0 \), and \( t_{i,j} > 0 \). If all directors report truthfully, \( \hat{t}_n = t_n \) then the consensus is \( a^* = (0, 0) \). Director 3 would benefit by misrepresentation: if director 3 pretended to have target \( \tilde{t}_3 = ((1 + \beta^{-1}) t_{21}, t_{22}) \), the new consensus would be \( \tilde{a} = (t_{21}, 0) \), at distance \( \|t_3 - \tilde{a}\| = t_{22} \), which is closer to \( t_3 \) than the original consensus \( a^* = 0 \), since \( \|t_3 - a^*\| = \sqrt{t_{21}^2 + t_{22}^2} \). Directors 1, 2, and 4 have similar incentives to deviate from the truthful reporting. Note that if \( t_{21} \) were zero, no agent would have an incentive to misrepresent preferences.

It may seem that the above example is a criticism of our definition of consensus. The following theorem shows that, in fact, there does not exist any decision rule that induces truthful reporting and satisfies natural assumptions about invariance under linear transformation (which are satisfied by our definition of consensus). Formally, a decision rule is a function \( A \) that maps any \( N \)-tuple of targets \( t_1, ..., t_N \), all located in the same action space \( \mathbb{R}^M \) for some \( M \), into the action space \( \mathbb{R}^M \). Here are the properties of \( A \) that we will use in our impossibility result.

**Definition 2.** The decision rule \( A \) is symmetric if, for all \( N \), \( M \), and \( t_1, ..., t_N \), every perturbation\(^\text{16}\) \( \pi \) of the agents 1 through \( N \) does not change the solution: \( A(t_{\pi(1)}, ..., t_{\pi(N)}) = A(t_1, ..., t_N) \) given that \( b_n = B \) for \( n = 1, ..., N \).

**Definition 3.** The decision rule \( A \) is invariant if, for all \( N \), \( M \), and \( t_1, ..., t_N \), \( A(a + Bt_1, ..., a + Bt_{\pi(N)}) = a + BA(t_1, ..., t_N) \), for all vectors \( a \in \mathbb{R}^M \) and all \( M \times M \) matrices \( B \) whose eigenvalues \( \lambda_1, ..., \lambda_M \) all have the same norm.

An invariant decision rule produces a decision that is invariant under rotation, translation, rescaling, and flipping.\(^\text{17}\) This a sensible requirement because the

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\(^\text{16}\)A perturbation \( \pi \) of 1 through \( N \) is a one-to-one mapping of \( \{1, ..., N\} \) to itself.

\(^\text{17}\)The vector \( a \) permits an arbitrary translation. The eigenvalues of \( B \) may be real or complex. Requiring all eigenvalues of \( B \) to have norm 1 would permit arbitrary rotations and flipping (change of orientation). Requiring all eigenvalues to have the same norm also permits rescaling.
agents’ ordinal preferences are invariant under these transformations. We do not require, for example, invariance under rescaling on one dimension but not others, which would not leave ordinal preferences unchanged.

**Definition 4.** The decision rule $A$ is devious if, for any $N$, $M$, $t_1, ..., t_N$, and $n$, there exists $t'_n$ such that

$$\|A(t_1, ..., t_n, ..., t_N) - t_n\| > \|A(t_1, ..., t'_n, ..., t_N) - t_n\|.$$

**Theorem 7.** All symmetric and invariant decision rules are devious.

*Proof.* Suppose, to the contrary, there is a symmetric and invariant decision rule $A$ that is not devious. Consider a board with three directors whose targets are $t_1$, $t_2$, and $t_3$ in $\mathbb{R}^2$ and bargaining power is $b_n = 1$, $n = 1, 2, 3$. First we show that, by symmetry and invariance, the decision is the midpoint $(t_1 + t_2 + t_3)/2$ whenever $t_1$, $t_2$, and $t_3$ are the vertices of an equilateral triangle. Consider the transformation $a + Bt$ that maps $t_1$ into $t_2$, $t_2$ into $t_1$, and $t_3$ into itself. By invariance, this transformation maps the decision $a^*$ into $a + Ba^*$. However, by symmetry the decision is the same as in the original problem: $a^* = a + Ba^*$. This implies that $a^*$ is on the line through $t_3$ and the midpoint between $t_1$ and $t_2$. Similarly, we can show that $a^*$ is on the line through $t_2$ and the midpoint between $t_1$ and $t_3$. Therefore, $a^*$ must be the midpoint of the triangle, $a^* = (t_1 + t_2 + t_3)/2$.

Next, we use a revealed preference argument to characterize the decision for a number of cases. Suppose that there exists $\hat{a} \in \text{Hull}(t_1, t_2, t_3)$ such that $\angle t_1 \hat{a} t_2 = \angle t_2 \hat{a} t_3 = \angle t_1 \hat{a} t_3 = 120^\circ$. Then we will show that the decision is $\hat{a}$. Consider the new set of vertices defined by

$$\hat{t}_n = \hat{a} + \frac{\max_j \|t_j - \hat{a}\|}{\|t_n - \hat{a}\|} (t_n - \hat{a}).$$

Because the $\hat{t}_n$s are at equal angles and equidistant from $\hat{a}$, they form an equilateral triangle with $\hat{a}$ at the center. Therefore, $\hat{a}$ would be the decision for agents with targets $\hat{t}_1$, $\hat{t}_2$, and $\hat{t}_3$. If $\hat{t}_1 \neq t_1$, changing $\hat{t}_1$ to $t_1$ cannot change the decision, since any other action that would be weakly preferred to $\hat{a}$ for an agent with target $t_1$ would be strictly preferred by an agent with target $\hat{t}_1$. Therefore, $\hat{a}$ is also the decision for agents with targets $t_1$, $\hat{t}_2$, and $\hat{t}_3$. Applying the same argument to agents 2 and 3 implies that $\hat{a}$ is also the
decision for agents $t_1$, $t_2$, and $t_3$, as claimed. A similar argument shows that a vertex $t_1$ such that $\angle t_2 t_1 t_3 = 120^\circ$ is the decision.

We are ready to construct the example that will lead to a contradiction. Let $t_1 = (1/2, \sqrt{3}/2)$, $t_2 = (0, 0)$, $t_3 = (-\sqrt{3}, 0)$. Since $\angle t_1 t_2 t_3 = 120^\circ$, the decision for these directors is $A(t_1, t_2, t_3) = t_2 = (0, 0)$. Let $\hat{t}_1 = \hat{a} + (1, \tan(67.5^\circ))$ and let $\hat{a} = ((\sqrt{2} - \sqrt{3})/2, (\sqrt{2} - 1)/2)$. It is straightforward to verify that $\hat{a} \in \text{HULL}(t_1, t_2, t_3)$ and $\angle \hat{t}_1 \hat{a} t_2 = \angle t_2 \hat{a} t_3 = \angle \hat{t}_1 \hat{a} t_3 = 120^\circ$.

Therefore, $A(\hat{t}_1, t_2, t_3) = \hat{a}$. Furthermore, $\|\hat{a} - t_1\| = \sqrt{3 + \sqrt{3} - \sqrt{6} - \sqrt{2}} < 1$. Thus, $\|A(\hat{t}_1, t_2, t_3) - t_1\| < \|A(t_1, t_2, t_3) - t_1\| = \|t_1\| = 1$, which implies that $A$ is devious.

If the targets are not publicly known and agents choose what targets to report, multiple reporting equilibria are possible. For example, given the true targets $t_1, \ldots, t_N$, it is an equilibrium if at least $N_0$ directors pretend their targets are $\hat{t}_n = a$, where $a$ is an arbitrary action, and $N_0 \geq (N + 3)/2$.

6 Conclusion

We have developed a simple model of board decision-making in the presence of diverse directors, with applications to director independence.

Appendix

The order of proofs in the Appendix differs from the order in which the results are presented in the body of the paper. Because some results in Sections 2 and 3 rely on the properties of consensus properties in Theorems 2–6, the Appendix offers proofs of these theorems first and then turns to the results of Sections 2 and 3.
Given $t_1, \ldots, t_N$, define function $f(a)$ on the action space $a \in \mathbb{R}^M$ as

$$f(a) = \sum_{n=1}^{N} b_n \| t_n - a \|. \quad (23)$$

**Lemma 5.** Function $f(a)$ given by (23) is convex. Moreover, if not all $t_1, \ldots, t_N$ are collinear, it is strictly convex.

**Proof.** For any two actions $a_1$ and $a_2$,

$$f(\beta a_1 + (1 - \beta)a_2) = \sum_{n=1}^{N} b_n \| t_n - \beta a_1 - (1 - \beta)a_2 \|$$

$$= \sum_{n=1}^{N} b_n \| \beta(t_n - a_1) + (1 - \beta)(t_n - a_2) \|$$

$$\leq \sum_{n=1}^{N} b_n (\| \beta(t_n - a_1) \| + \| (1 - \beta)(t_n - a_2) \|)$$

$$= \beta \sum_{n=1}^{N} b_n \| t_n - a_1 \| + \sum_{n=1}^{N} b_n (1 - \beta) \| t_n - a_2 \|$$

$$= \beta f(a_1) + (1 - \beta)f(a_2). \quad (24)$$

Therefore, $f(a)$ is convex.

Function $f(a)$ is strictly convex if and only if (24) is a strict inequality for every $a_1$ and $a_2$. Notice that (24) holds with equality if and only if $(t_n - a_1)$ and $(t_n - a_2)$ are collinear for all $n$. Collinearity of $(t_n - a_1)$ and $(t_n - a_2)$ implies that

$$t_n - a_1 = \beta_n(t_n - a_2).$$

Rearranging, find that $t_n$ is a weighted average of $a_1$ and $a_2$:

$$t_n = \frac{1}{1 - \beta_n} a_1 + \frac{\beta_n}{1 - \beta_n} a_2$$

for all $n$. Hence, collinearity of $(t_n - a_1)$ and $(t_n - a_2)$ implies that $t_1, \ldots, t_N$ are collinear. Therefore, (24) is a strict inequality if $t_1, \ldots, t_N$ are not collinear. 

$\square$

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Proof of Theorem 1: Characterization

Proof. According to Lemma 5, $f(a)$ is convex. Therefore, the action $a^*$ minimizes $f(a)$ if and only if it satisfies the first-order condition. Because $\sum b_n e_n(a)$ is the subgradient correspondence for $f(a)$ (as we show below), the first-order condition is given by (1).

The subgradient correspondence for $f(a)$ can be derived as follows. Let $g_n(a) = \|t_n - a\|$. If $a \neq t_n$, then the gradient of $g_n(a)$ at $a$ is

$$
\frac{d}{da}g_n(a) = \frac{d}{da}\|t_n - a\| = -\frac{t_n - a}{\|t_n - a\|}.
$$

If $a = t_n$, then $\varepsilon \in dg(a)$ if and only if

$$
\varepsilon' \delta \leq g(t_n + \delta) - g(t_n) = \|t_n - t_n - \delta\| - \|t_n - t_n\| = \|\delta\|.
$$

The above holds if and only if $\|\varepsilon\| \leq 1$, or in other words, if and only if $\varepsilon \in e(a)$. Hence, for any $a$, $e(a)$ is the subgradient correspondence for $g(a) = \|t_n - a\|$. Therefore, $\sum b_n e_n(a)$ is the subgradient correspondence for $f(a)$. \qed

Proof of Theorem 2: Existence and Uniqueness

Proof. Let $S$ denote the ball in the action space that is centered at zero and has a radius of $(\frac{2}{B} \sum b_n \|t_n\|)$, where $B = \sum b_n$; therefore $S \equiv \{a \in \mathbb{R}^M : \|a\| \leq \frac{2}{B} \sum b_n \|t_n\|\}$. Because $f(a)$ is continuous, there exists an action $a^*$ that minimizes $f(a)$ on the set $S$, $a^* = \text{argmin}_{a \in B} f(a)$. To prove existence, it remains to show that $a^*$ is a global minimum of $f(a)$.

Because $a^*$ minimizes $f(a)$ on set $S$, it in particular implies that

$$
\begin{align*}
(25) \quad f(a^*) &\leq f(0) = \sum_{n=1}^{N} b_n \|t_n - 0\| = \sum_{n=1}^{N} b_n \|t_n\| \\
\text{For the actions } a \text{ outside ball } S, a \notin S, \\
(26)
\end{align*}
$$
\[ f(a) = \sum_{n=1}^{N} b_n\|t_n - a\| \geq \sum_{n=1}^{N} b_n (\|a\| - \|t_n\|) > \sum_{n=1}^{N} b_n \left( \frac{2}{B} \sum_{k=1}^{N} b_k \|t_k\| - \|t_n\| \right) \]
\[ = 2 \sum_{k=1}^{N} b_k \|t_k\| - \sum_{n=1}^{N} b_n \|t_n\| = \sum_{n=1}^{N} b_n \|t_n\|, \]

where the first inequality follows from the triangle inequality, and the second follows because \( a \notin S \) implies \( \|a\| > \frac{2}{B} \sum_{k=1}^{N} \|t_k\| \). Combining (25) and (26) obtains that \( f(a^*) < f(a) \) for all \( a \). In other words, \( a^* \) is a global minimum.

From Lemma 5, the function \( f(a) \) is strictly convex when not all \( t_1, \ldots, t_N \) are collinear, and thus has a unique minimum. Therefore, the solution to (6) is unique if not all \( t_1, \ldots, t_n \) are collinear.

\[ \square \]

Proof of Theorem 4: Majority Rule

Proof. Let \( B_0 = \sum_{n=1}^{N_0} b_n \) and \( B_1 = \sum_{n=N_0+1}^{N} b_n \). Define vectors \( z_n \) for \( n = 1, \ldots, N \) as follows:
\[ z_n = \begin{cases} \frac{t_n-a}{\|t_n-a\|} & \text{if } N_0 + 1 \leq n \leq N \\ \frac{1}{B_0} \sum_{k=N_0+1}^{N} b_k z_k & \text{if } n \leq N_0. \end{cases} \]

These vectors \( z_n \) satisfy (1). Therefore, from Theorem 1, \( t \) solves problem (6). From Theorem 2, the consensus is unique when not all \( t_1, \ldots, t_N \) are collinear.

To prove the theorem, it remains to show that when all \( t_1, \ldots, t_N \) are collinear, the consensus is unique when \( B_0 > B_1 \). Indeed, for any action \( a \neq t \), let \( \hat{z}_n \in e_n(a) \) and observe that
\[ \| \sum_{n=1}^{N} \hat{b}_n z_n \| \geq \| B_0 \| - \| \sum_{n=N_0+1}^{N} b_n \hat{z}_n \| \geq B_0 - B_1 > 0. \]

Therefore, from Theorem 1, \( a \) does not solve problem (6).

\[ \square \]

Proof of Theorem 5: Median Voter

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Proof. We show first that any consensus \( a \) is collinear with \( t_1, \ldots, t_N \): \( a = a_0 + \beta(a_1 - a_0) \). Suppose that, to the contrary, \( a = a_0 + \beta(a_1 - a_0) + x \) is a consensus, where \( x \) is a non-zero vector, orthogonal to \( a_1 - a_0 \): \( x'(a_1 - a_0) = 0 \).

Let \( \hat{a} = a_0 + \beta(a_1 - a_0) \). Then, for any \( n, k \)

\[
\sum_{n:a \neq t_n} b_n z_n = \sum_{n:a \neq t_n} b_n \frac{t_n - a}{\|t_n - a\|} = \sum_{n:a \neq t_n} b_n \frac{a_1 - a_0}{\|a_1 - a_0\|} \frac{\rho_n - \beta}{\|\rho_n - \beta\|} = \frac{a_1 - a_0}{\|a_1 - a_0\|} \sum_{n:a \neq t_n} b_n \frac{\rho_n - \beta}{\|\rho_n - \beta\|} = \frac{a_1 - a_0}{\|a_1 - a_0\|} (b_g N_g - b_l N_l).
\]

Action \( a = a_0 + \beta(a_1 - a_0) \) is a consensus if and only if \( |b_g N_g - b_l N_l| \leq b_e N_e \), which is equivalent to

\[
\sum_{n=1}^N b_n N = b_l N_l + b_g N_g + b_e N_e \geq b_l N_l + b_g N_g + |b_g N_g - b_l N_l| = 2 \max\{b_l N_l, b_g N_g\}.
\]

Hence, \( a \) is a consensus if and only if \( \max\{N_l, N_g\} \leq N_0 \). When (20) holds with inequality, this implies that \( \beta = \rho_{N_0} \). Otherwise, \( \beta \in [\rho_{N_0-1}, \rho_{N_0}] \).

Proof of Theorem 6: Extreme Director

Proof. The bound is satisfied trivially if \( a^* \in H \). Suppose that \( a^* \notin H \). Let
Let \( t^* \) be the projection of \( a^* \) onto \( H \). Because \( a^* \) solves (6),
\[
\frac{\partial}{\partial \beta} \bigg|_{\beta=0} \left( \sum_{n=1}^{N} \left\| t_n - \left( a^* + \beta \frac{t^* - a^*}{\left\| t^* - a^* \right\|} \right) \right\| \right) = \sum_{n=1}^{N} \left( \frac{(t_n - a^*)(t^* - a^*)}{\left\| t_n - a^* \right\| \left\| t^* - a^* \right\|} \right) = 0.
\]

Note that \( (\partial/\partial \beta)_{\beta=0} \left( t_N - (a^* + \beta(t^* - a^*)/\left\| t^* - a^* \right\|) \right) \geq -1 \). Thus,
\[
(27) \quad \frac{\partial}{\partial \beta} \bigg|_{\beta=0} \left( \sum_{n=1}^{N-1} \left\| t_n - \left( a^* + \beta \frac{t^* - a^*}{\left\| t^* - a^* \right\|} \right) \right\| \right) \leq 1.
\]

For \( n < N \),
\[
\frac{\partial}{\partial \beta} \bigg|_{\beta=0} \left( t_n - \left( a^* + \beta \frac{t^* - a^*}{\left\| t^* - a^* \right\|} \right) \right) = \frac{(t_n - t^* + t^* - a^*)(t^* - a^*)}{\left\| t^* - a^* \right\| \left\| t_n - a^* \right\|} = \frac{\left\| t^* - a^* \right\|}{\left\| t_n - a^* \right\|} + \frac{(t_n - t^*)(t^* - a^*)}{\left\| t^* - a^* \right\| \left\| t_n - a^* \right\|}
\]

Since \( t^* \) is the projection of \( a^* \) on \( H \), \( t^* - a^* \) separates \( t^* \) from \( H \) so \( t_n \in H \) implies \((t_n - t^*)(t^* - a^*) \geq 0 \). From a triangle inequality,
\[
\left\| t_n - a^* \right\| \leq \left\| t_n - t^* \right\| + \left\| t^* - a^* \right\|
\]
\[
= \left\| t_n - \sum_{j=1}^{N} w_j t_j \right\| + \left\| t^* - a^* \right\|
\]
\[
= \sum_{j=1}^{N} w_j \left\| t_n - t_j \right\| + \left\| t^* - a^* \right\|
\]
\[
\leq \sum_{j=1}^{N} w_j \left\| t_n - t_j \right\| + \left\| t^* - a^* \right\|
\]
\[
\leq K + d(H, a^*)
\]

\footnote{The formula in the text assumes that the derivative exists. We know that the distance measure is not differentiable when \( a^* = t_n \) for some \( n \). For \( n < N \), \( a^* \neq t_n \) because \( a^* \notin H \). In the event that \( a^* = t_N \), substitute the relevant member of the derivative correspondence and the rest of the proof goes through.}
Therefore,
\[
\frac{\partial}{\partial \beta} \left| \left| t_n - \left( a^* + \beta \frac{t^* - a^*}{\|t^* - a^*\|} \right) \right| \right| \geq \frac{d(H, a^*)}{K + d(H, a^*)}
\]
Substituting the above into (27), obtain
\[
(N - 1) \frac{d(H, a^*)}{K + d(H, a^*)} \leq 1
\]
Rearranging the above, obtain that the claimed bound (21) on the distance holds.

\textbf{Proof of Lemma 1: (Extreme Director)}

By Theorem 1, \( a_0^* = \bar{y} \) is the consensus for the original \( N = N_0 \) directors because, taking \( z_n = (t_n - a_0^*)/\|t_n - a_0^*\| \),

\begin{equation}
\sum_{n=1}^{N_0} z_n = \sum_{n=1}^{N_0} \frac{t_n - a_0^*}{\|t_n - a_0^*\|} = \sum_{n=1}^{N_0} \frac{r(0, \sin(2\pi n/N_0), \cos(2\pi n/N_0))}{r} = 0,
\end{equation}

where the last equality holds because the points are equally spaced on a circle. Formally, recall that \( e^{i\theta} = \cos(\theta) + i\sin(\theta) \) and observe that
\[
\sum_{n=1}^{N_0} e^{i2\pi n/N_0} = \sum_{n=1}^{N_0} \left( e^{i2\pi /N_0} \right)^n = \frac{e^{i2\pi /N_0} - 1}{e^{i2\pi /N_0} - 1} = \frac{e^{i2\pi} - 1}{e^{i2\pi} - 1} = \frac{\cos(2\pi) + i\sin(2\pi) - 1}{e^{i2\pi} - 1} = 0.
\]
Note that, by construction, \( z_n \in e_n(a^0) \), where \( e_n(a) \) is defined in (2).

For \( N = N_0 + 1 \) directors, \( a^* = t_{N_0+1} \) when \( b_{N_0+1} \geq N_0 \) follows from the majority rule result (Theorem 4). If \( b_{N_0+1} < N_0 \), let \( a^* = \bar{y}\lambda + (h_0, 0, 0) \), where, as claimed in the statement of the lemma, \( h_0 = \min\{h, rb_{N_0+1}/\sqrt{N_0^2 - b_{N_0+1}^2}\} \).

We want to apply Theorem 1 to show that \( a^* \) is the consensus for the \( N = N_0 + 1 \) directors. We take \( z_n = (t_n - a^*)/\|t_n - a^*\| \) for \( n \leq N_0 \) and \( z_{N_0+1} = (N_0h_0/b_{N_0+1}\sqrt{r^2 + h_0^2}, 0, 0) \). Then,

\[
(29) \sum_{n=1}^{N_0+1} b_n z_n = \sum_{n=1}^{N_0} \frac{t_n - a^*}{\|t_n - a^*\|} + b_{N_0+1} z_{N_0+1} = \frac{\|t_n - \bar{y}\lambda\|}{\|t_n - a^*\|} \sum_{n=1}^{N_0} \frac{t_n - \bar{y}\lambda}{\|t_n - a^*\|} \frac{N_0(a^* - \bar{y}\lambda)}{\|t_n - a^*\|} + b_{N_0+1} z_{N_0+1} = -\frac{N_0}{\|t_n - a^*\|}(a^* - \bar{y}\lambda) + b_{N_0+1} z_{N_0+1} = -\frac{N_0}{\sqrt{r^2 + h_0^2}}(h_0, 0, 0) + \left( \frac{N_0h_0}{\sqrt{r^2 + h_0^2}}, 0, 0 \right),
\]

where third equality is obtained using derivation similar to (28). Note that, by construction, \( z_n \in e_n(a^*) \) for \( n \leq N_0 \). If \( h > rb_{N_0+1}/\sqrt{N_0^2 - b_{N_0+1}^2} \), then \( h_0 = rb_{N_0+1}/\sqrt{N_0^2 - b_{N_0+1}^2} \) and \( z_{N_0+1} = (1, 0, 0) \in e_{N_0+1}(a^*) \). If \( h \leq rb_{N_0+1}/\sqrt{N_0^2 - b_{N_0+1}^2} \), then \( h_0 = h \), and

\[
\|z_{N_0+1}\|^2 = \frac{N_0^2 h_0^2}{b_{N_0+1}(r^2 + h_0^2)} = \frac{N_0^2}{b_{N_0+1}^2}(1 - \frac{r^2}{r^2 + h_0^2}) \leq \frac{N_0^2}{b_{N_0+1}^2} \left( 1 - \frac{r^2}{r^2 + r^2b_{N_0+1}/(N_0^2 - b_{N_0+1}^2)} \right) = 1,
\]

and again, \( z_{N_0+1} \in e_{N_0+1}(a^*) \).
Proof of Lemma 3: Ineffective Majority of Independent Directors

Without loss of generality, let the new director’s ideal be
\[ t_{N_o+N_i+1} = (\delta + \gamma \cos(\theta), \gamma \sin(\theta), 0, 0, 0). \]

If \( \gamma \neq 0 \), from Theorem 1, the consensus with the new director is at \( a^* = (\delta, 0, 0, 0, 0) \) if
\[
\sum_{n=1}^{n=N_o} \frac{t_n - a^*}{\|t_n - a^*\|} + \frac{t_{N_o+N_i+1} - a^*}{\|t_{N_o+N_i+1} - a^*\|} = N_i\varepsilon, 
\]
where \( \|\varepsilon\| \leq 1 \), or equivalently, if
\[
\left(-\frac{N_o\delta}{\sqrt{\delta^2 + r^2}} + \cos(\theta)\right)^2 + (\sin(\theta))^2 \leq N_i^2. 
\]

Noting that \( \cos(\theta)^2 + (\sin(\theta))^2 = 1 \) and rearranging, obtain
\[
\frac{N_o\delta}{2\sqrt{\delta^2 + r^2}} - \frac{(N_i^2 - 1)\sqrt{\delta^2 + r^2}}{2N_o\delta} \leq \cos(\theta). 
\]

The sufficient condition (14) then follows from \( \cos(\theta) \leq 1 \)

References


Cai, Hongbin, 2004, Costly Participation and Heterogeneous Preferences in Informational Committees, working paper.
Gomes, Armando and Walter Novaes, 2005, Sharing of Control versus Monitoring as Corporate Governance Mechanisms, working paper.


