The Fallacy of Large Numbers and A Defense of Diversified Active Managers

Philip H. Dybvig
Washington University in Saint Louis

First Draft: March 10, 2003
This Draft: March 27, 2003

ABSTRACT

Traditional mean-variance calculations tell us that the return to a well-diversified portfolio of stocks with an average beta of one will be close to the benchmark. This may suggest that an active manager seeking to beat the market cannot succeed with a large diversified portfolio, and that any successful active portfolio must be limited to a small number of stocks. However, the traditional calculations are not appropriate for an active manager, since traditional calculations assume a fixed portfolio, not an active portfolio that varies depending on the information acquired by the manager. For an active manager, a well-diversified portfolio can add trading profits to the market portfolio without adding significant idiosyncratic risk to the return. This implies a defense of holding large active portfolios, which can potentially provide superior performance and do not expose investors to idiosyncratic risk even if the manager has no talent.

Should active managers hold large portfolios or small portfolios? One common argument says that active managers with large well-diversified portfolios are doomed to failure because the power of diversification implies their returns will be close to a benchmark portfolio.[1] However, the traditional mean-variance analysis of diversification is incompatible with skilled active management, because the traditional analysis assumes the manager does not have any superior ability. This paper modifies the traditional analysis to include the possibility of superior performance, and finds that a well-diversified portfolio can exhibit superior performance. Therefore, there is no logical reason why active managers holding large portfolios cannot be successful.

Security Returns and Information Consider a manager who has a skill in choosing between pairs of stocks. There are N pairs of stocks, and we index stocks by a pair (n,i), where $i=1$ or 2 and n runs from 1 to N. Based on research, our manager generates a private signal s_n for each pair n. The signal s_n can be interpreted as a part of what uninformed investors view as idiosyncratic risk but can be used by our manager to generate trading profits. Specifically, we assume that the return of security (n,i) is given by the market model

$$(1) \qquad R_{n,i} = \begin{cases} R^f + \beta_n(R^B - R^f) + s_n + \varepsilon_{n,1} & i=1 \\ R^f + \beta_n(R^B - R^f) - s_n + \varepsilon_{n,2} & i=2 \end{cases}$$

where R^f is the riskfree rate, β_n is the beta coefficient measuring exposure to market risk, R^B is the return to the benchmark (or market) portfolio, s_n is a signal observed by our informed manager, and $\varepsilon_{n,i}$ is the idiosyncratic risk term for security (n,i) (or more precisely the part of idiosyncratic risk that is not known in advance by our manager). Note that we are assuming that both stocks in pair n have the same level β_n of market risk exposure. We will assume that the average beta coefficient across pairs is one:

$$(2) \qquad \frac{1}{N}\sum_{n=1}^{N} \beta_n = 1.$$

[1] We are assuming that active managers are trying to beat the benchmark. Of course, matching the benchmark would be an improvement for many actual managers!

This is sensible given that the benchmark, which is supposed to be representative of stocks in our universe, always has a beta of one.[2] To keep the calculations simple, let

$$(3) \qquad s_n = \begin{cases} \delta & \text{with probability } 1/2 \\ -\delta & \text{with probability } 1/2 \end{cases}$$

for some $\delta > 0$. In this case, s_n, the part of the idiosyncratic noise of asset (n,i) that the informed agent knows about, has mean zero and variance $E[(s_n - 0)^2] = E[\delta^2] = \delta^2$. We also assume the $\varepsilon_{n,i}$'s, the unknowable parts of the idiosyncratic noise terms, all have the same standard deviation $\sigma > 0$ (and therefore variance $\sigma^2 > 0$).[3]

Traditional Analysis: Uninformed Manager Consider a fixed portfolio placing equal amounts in one stock of each of the pairs. Let $I(n)$ be the security chosen from the nth pair; $I(n) = 1$ if the portfolio chooses the first security in the nth pair, or $I(n) = 2$ if the portfolio chooses the second security in the nth pair. Further, let

$$(4) \qquad s_n^* \equiv \begin{cases} s_n & I(n) = 1 \\ -s_n & I(n) = 2 \end{cases}$$

be the associated signal term. Then the return on our portfolio is

$$(5) \qquad \begin{aligned} R^{fixed} &= \sum_{n=1}^{N} \frac{1}{N} R_{n,I(n)} \\ &= \sum_{n=1}^{N} \frac{1}{N} (R^f + \beta_n (R^B - R^f) + s_n^* + \varepsilon_{n,I(n)}) \end{aligned}$$

[2]The average beta is one almost by definition if the benchmark is an equal-weighted portfolio. If the benchmark is a value-weighted portfolio, then the value-weighted average beta would have to be one.

[3]We could generalize the analysis to use value-weighted portfolios and give individual assets differing levels of idiosyncratic risk. Under these assumptions, convergence of the risk term to zero would be slower, but would still occur provided no asset remains large compared to the economy as we add assets and idiosyncratic variances are uniformly bounded.

$$
\begin{aligned}
&= \left(\frac{1}{N}\sum_{n=1}^{N} R^f\right) + \left(\frac{1}{N}\sum_{n=1}^{N}\beta_n\right)(R^B - R^f) + \left(\sum_{n=1}^{N}\frac{1}{N}(s_n^* + \varepsilon_{n,I(n)})\right) \\
&= R^f + (R^B - R^f) + \frac{1}{N}\sum_{n=1}^{N}\left(s_n^* + \varepsilon_{n,I(n)}\right) \\
&= R^B + \frac{1}{N}\sum_{n=1}^{N}(s_n^* + \varepsilon_{n,I(n)}).
\end{aligned}
$$

For this fixed portfolio, the mean deviation from the market return

$$
\begin{aligned}
(6) \qquad E[R^{fixed} - R^B] &= E\left[\frac{1}{N}\sum_{n=1}^{N}(s_n^* + \varepsilon_{n,I(n)})\right] \\
&= 0,
\end{aligned}
$$

because all the s_n^*'s and $\varepsilon_{n,I(n)}$'s have mean zero. Further, because the s_n^*'s and $\varepsilon_{n,I(n)}$'s are all jointly independent, the variance of this deviation does not include any cross (covariance) terms. Therefore, the variance of the deviation from the market is

$$
\begin{aligned}
(7) \qquad \mathrm{var}(R^{fixed} - R^B) &= E\left[\left(\frac{1}{N}\sum_{n=1}^{N}(s_n^* + \varepsilon_{n,I(n)}) - 0\right)^2\right] \\
&= \left(\frac{1}{N}\right)^2 \sum_{n=1}^{N}(\delta^2 + \sigma^2) \\
&= \left(\frac{1}{N}\right)^2 (\delta^2 + \sigma^2)N \\
&= \frac{1}{N}(\delta^2 + \sigma^2).
\end{aligned}
$$

As N increases, the variation of the portfolio's deviation from the market falls towards zero. Having small variance puts definite bounds on the probability of significant deviations. For example, one conservative bound based on the Chebyshev inequality[4] says that the probability of exceeding the market some fixed amount x can be bounded by a number that gets small as N gets large:

$$\begin{aligned} \text{prob}(R^{fixed} - R^B > x) &< \frac{\text{var}(R^{fixed} - R^B)}{x^2} \\ &= \frac{\delta^2 + \sigma^2}{Nx^2}. \end{aligned}$$

Here, $1/N$ multiplies a constant, so the probability of doing significantly better (defined by any chosen level $x > 0$) than the benchmark gets smaller and smaller as N increases (for fixed x). This is the traditional analysis that is interpreted as showing that a well-diversified portfolio cannot have significant superior performance.

What is wrong with the traditional analysis? Nothing, if the analysis is applied to a fixed portfolio. However, an active manager does not hold a fixed portfolio, and if the manager has ability, the portfolio chosen is correlated with what an uninformed investor would view as the error term.[5] For an active portfolio, the mean and the variance of the deviation are both different than what is computed in the traditional analysis.

New Analysis: Informed Manager However, the traditional analysis is incorrect if the manager is informed so that the portfolio is not fixed and is chosen instead as a function of the manager's information. For example, suppose that our manager follows the natural strategy of buying the first portfolio in the pair when it is favored (because $s_n = \delta$) and the second portfolio in the pair when it is favored (because $s_n = -\delta$). This will be the correct strategy if shorts are not allowed and

[4]Chebyshev's inequality says that for any random variable X with finite variance and any constant $a > 0$, $\text{prob}(|X| > a) \leq E[X^2]/a^2$. In particular, if X has mean zero, this implies that $\text{prob}(X > a) \leq \text{var}(X)/a^2$, since $\text{prob}(X > a) \leq \text{prob}(|X| > a)$ and $\text{var}(X) = E[X^2]$ when $E[X] = 0$. You may have seen the inequality under another phonetically similar name; there is a lot of variation in the conversion from Cyrillic to English spelling of Chebyshev's name.

[5]Our analysis concerns a manager whose ability is in picking stocks, not in timing the market. A talented market timer can make money using a well-diversified portfolio whose leverage is changing over time. That is not the most interesting case for the current analysis. Models of market timing (as well as stock selection), including an analysis of the pitfalls of using traditional performance measures to evaluate a market timer, are presented by Dybvig and Ross [1985].

N is large enough to diversify tracking error to the manager's satisfaction. Let $\iota(n)$ be the choice from pair n, i.e.,

$$(8) \qquad \iota(n) \equiv \begin{cases} 1 & \text{if } s_n = \delta \\ 2 & \text{if } s_n = -\delta \end{cases}$$

Then the return on the active manager's portfolio is

$$
\begin{aligned}
(9) \qquad R^{active} &= \sum_{n=1}^{N} \frac{1}{N} R_{n,\iota(n)} \\
&= \sum_{n=1}^{N} \frac{1}{N} (R^f + \beta_n (R^B - R^f) + \delta + \varepsilon_{n,\iota(n)}) \\
&= \left(\frac{1}{N} \sum_{n=1}^{N} R^f \right) + \left(\frac{1}{N} \sum_{n=1}^{N} \beta_n \right) (R^B - R^f) + \delta + \left(\sum_{n=1}^{N} \frac{1}{N} \varepsilon_{n,\iota(n)} \right) \\
&= R^f + (R^B - R^f) + \delta + \frac{1}{N} \sum_{n=1}^{N} \varepsilon_{n,\iota(n)} \\
&= R^B + \delta + \frac{1}{N} \sum_{n=1}^{N} \varepsilon_{n,\iota(n)}.
\end{aligned}
$$

The mean deviation from the benchmark is

$$
\begin{aligned}
(10) \qquad E[R^{active} - R^B] &= E\left[\delta + \frac{1}{N} \sum_{n=1}^{N} \varepsilon_{n,\iota(n)} \right] \\
&= \delta,
\end{aligned}
$$

since each $\varepsilon_{n,i}$ has mean zero and $\iota(n)$ is a function of s_n and is therefore drawn independently of each $\varepsilon_{n,i}$.[6] This means that the portfolio has superior performance of δ on average. For computing the variance, as in the traditional case, our

[6] More formally, $\varepsilon_{n,\iota(n)}$ can be written as $(s_n + \delta)\varepsilon n, 1/2\delta + (\delta - s_n)\varepsilon n, 2/2\delta$, as can be verified by noting this is true whether $s_n = \delta$ or $s_n = -\delta$. Then the result follows from the facts that s_n is drawn independently of the $\varepsilon_{n,i}$'s and the $\varepsilon_{n,i}$'s both have mean zero.

assumptions about independence of various random draws implies that there are no cross (covariance) terms:

$$
\begin{aligned}
(11)\quad \mathrm{var}(R^{active}-R^B) &= E\left[\left(\delta+\frac{1}{N}\left(\sum_{n=1}^{N}\varepsilon_{n,\iota(n)}\right)-\delta\right)^2\right] \\
&= \left(\frac{1}{N}\right)^2\sum_{n=1}^{N}\sigma^2 \\
&= \left(\frac{1}{N}\right)^2\sigma^2 N \\
&= \frac{1}{N}\sigma^2.
\end{aligned}
$$

As N increases, the variance of the deviation declines to zero, and consequently the superior performance of δ becomes more and more of a sure thing. In particular, for any $x>0$, Chebyshev's inequality implies that outperforming the benchmark by any less than $x-\delta$ becomes increasingly improbable as x increases:

$$
\begin{aligned}
(12)\quad \mathrm{prob}(R^{active}-R^B<\delta-x) &\leq \frac{\mathrm{var}(R^{active}-R^B-\delta)}{x^2} \\
&= \frac{\sigma^2}{Nx^2}
\end{aligned}
$$

Therefore, the probability of outperforming the benchmark by significantly less than δ gets smaller and smaller as the number of securities N increases.

A Paradox There may seem to be an inconsistency with the result (8) that for large N a passive portfolio is very unlikely to outperform the benchmark significantly and the result (12) that for large N the active portfolio is almost sure to outperform the benchmark significantly. After all, each portfolio the active manager might choose as a function of the signals is one of the passive portfolios for which (8) applies. For example, if we let $x=\delta/2$, then (8) would suggest that outperformance greater than $\delta/2$ is very unlikely, while (12) suggests that outperformance less than $\delta/2$ is very unlikely.[7] The two seem incompatible since the

[7] The argument in the text assumes that outperformance of exactly $\delta/2$ has probability zero,

probabilities should add to one.[8]

The answer to this apparent paradox is that a particular portfolio is chosen only a very small fraction ($= 1/2^N$) of the time and this is in situations when an unusually large return to the particular portfolio is expected. Since the probability of the portfolio being chosen is even smaller than the probability of an extreme return, there is no contradiction. That is why it is okay for the probability of performance of approximately δ given that this portfolio is selected can be so much larger than the probability of performance of approximately δ absent knowledge of the signals.

A Case for Large Portfolios The analysis above shows that there is no logical inconsistency between holding a large diversified portfolio and superior returns by an active manager. Indeed, a manager following this sort of policy could have some advantages over a manager choosing to hold a small number of stocks. For one, if the manager is not actually skilled, the portfolio return will be close to the index and therefore little damage from exposure to too much idiosyncratic risk will be done. Furthermore, having a large portfolio makes it easier to evaluate performance, since the degree of idiosyncratic risk to hide the performance is so much smaller.

Of course, this is just an example, and as such it shows that there is no logical impediment to superior performance. It remains an open question whether investing with an active manager holding a large portfolio is a good idea. Of course, many people are skeptical that any manager can obtain superior performance, and it is not known whether the shortage of concrete evidence of superior performance is due to lack of performance or the difficulty of distinguishing luck and skill. If superior performance is possible, perhaps the manager knows about only a few stocks, as might be the case for an analyst who specializes in analysis of a small set of industries. This type of manager would not be a good person to hire for active management of a diversified portfolio. On the other hand, a manager with a product for doing quantitative technical analysis to identify temporary liquidity needs might plausibly have a small amount of information about each of a

---

which is exactly true if the idiosyncratic noise terms have densities, and approximately true in general since $\sigma > 0$.

[8] The heuristic argument in the text assumes that outperformance of exactly $\delta/2$ has probability zero, which is exactly true if the idiosyncratic noise terms have densities, and approximately true in general (for both active and passive portfolios) for N large.

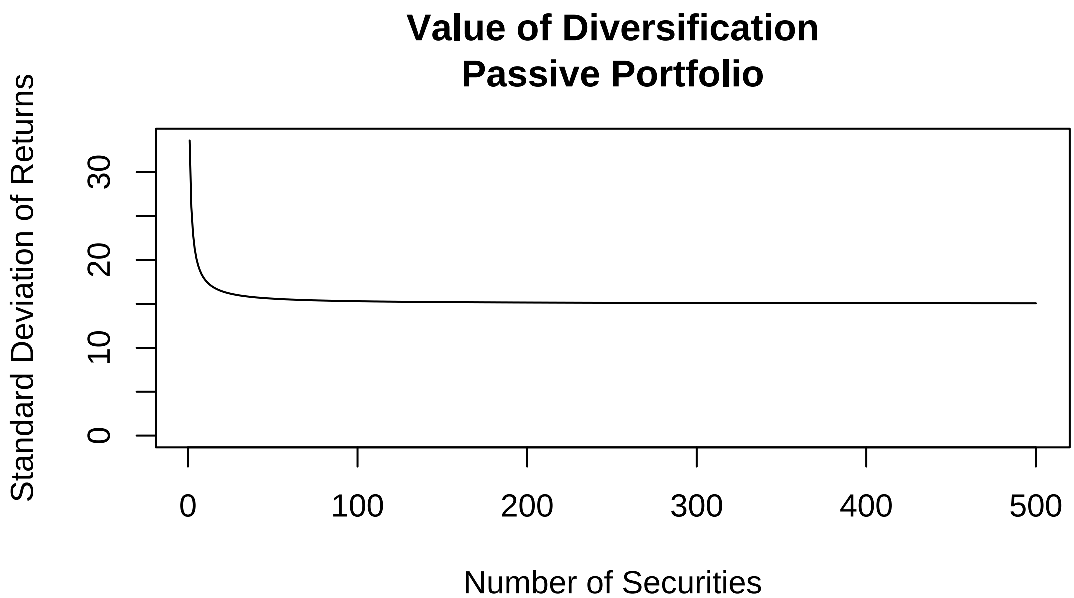

Figure 1: The value of diversification declines rapidly as we add securities, as is shown in this traditional figure.

large number of stocks and might be a likely candidate for superior performance. How likely are these scenarios to produce superior performance? Currently, this is a matter for judgment, and fortunate is the pension fund or endowment whose investments committee, by skill or luck, gets the answer right.

Numerical Results Figure 1 shows the traditional plot of the value of diversification in our context. This assumes the model (1) with one security chosen from each pair. We assume the benchmark has a standard deviation of 15%/year, that each stock's unknowable idiosyncratic risk has a standard deviation of $\sigma =$ 30%/year and that each stock's knowable idiosyncratic risk has a standard deviation $\delta = 2\%$/year. The graph is almost indistinguishable if we plot the standard deviation for our informed agent.

Figures 2 and 3 describe how the distribution of excess returns depends on the number of securities for the active and passive strategies. In these graphs, the solid line indicates the mean (and median) return, which is independent of the

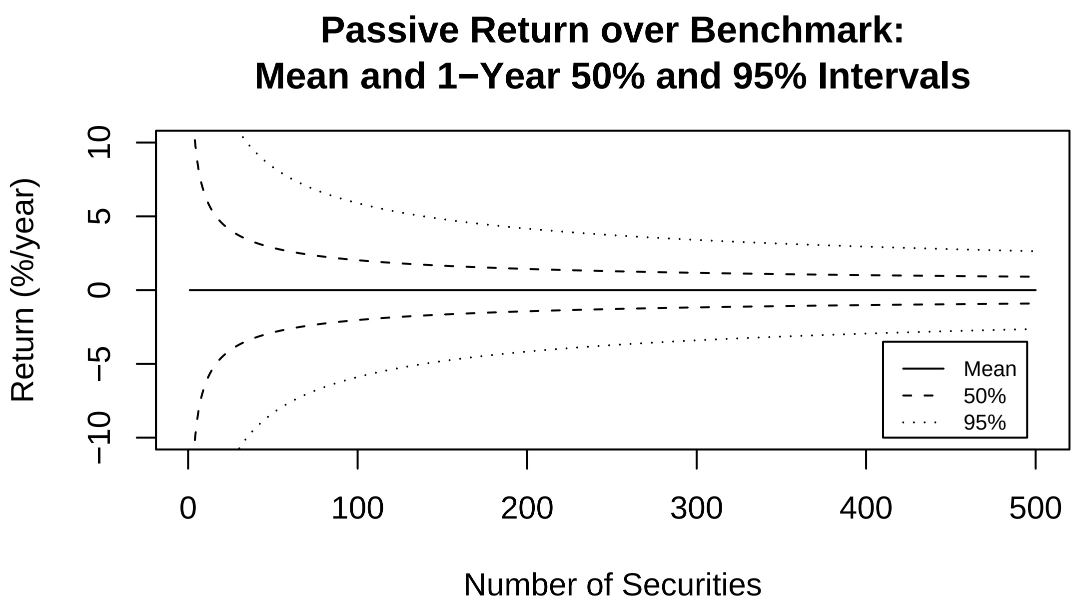

Figure 2: As we add securities, the passive portfolio's deviation from the benchmark converges slowly to zero.

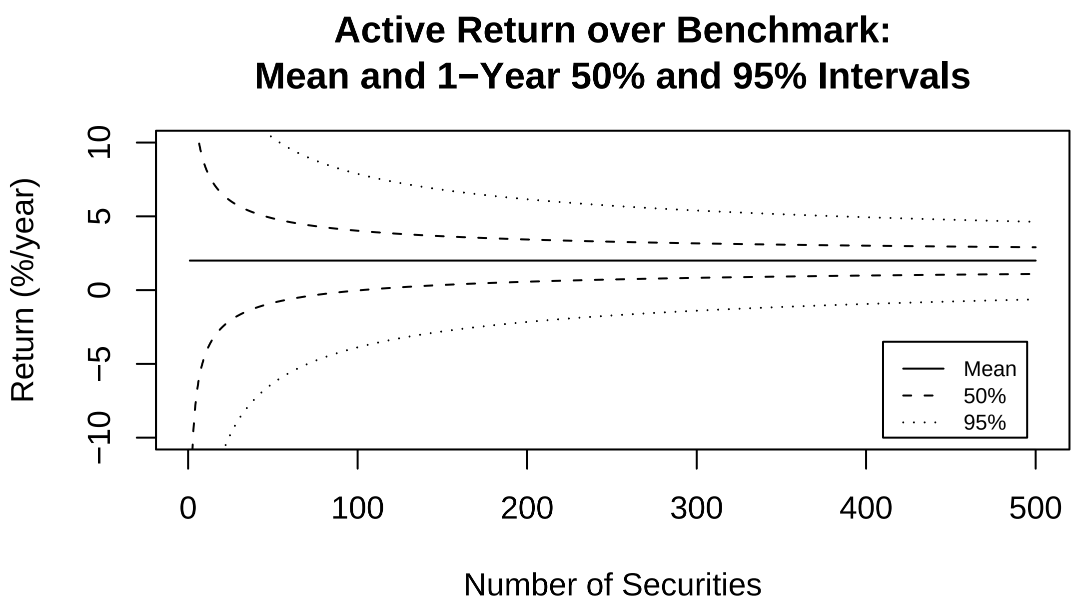

Figure 3: As we add securities, the active portfolio's deviation from the benchmark converges slowly to $+2\%$.

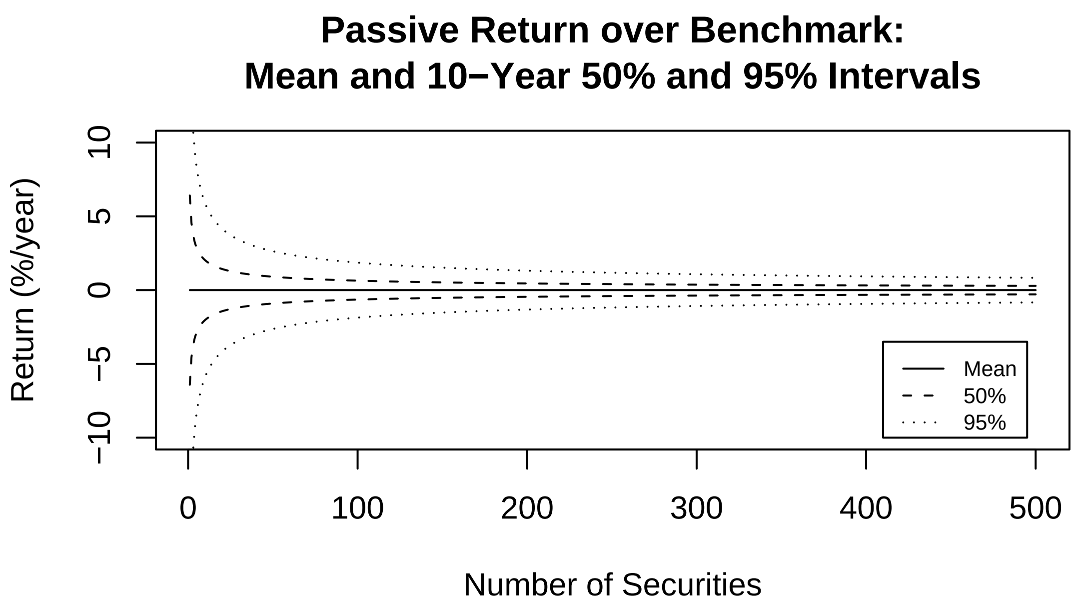

Figure 4: As we add securities, the passive portfolio's deviation from the benchmark converges slowly to zero.

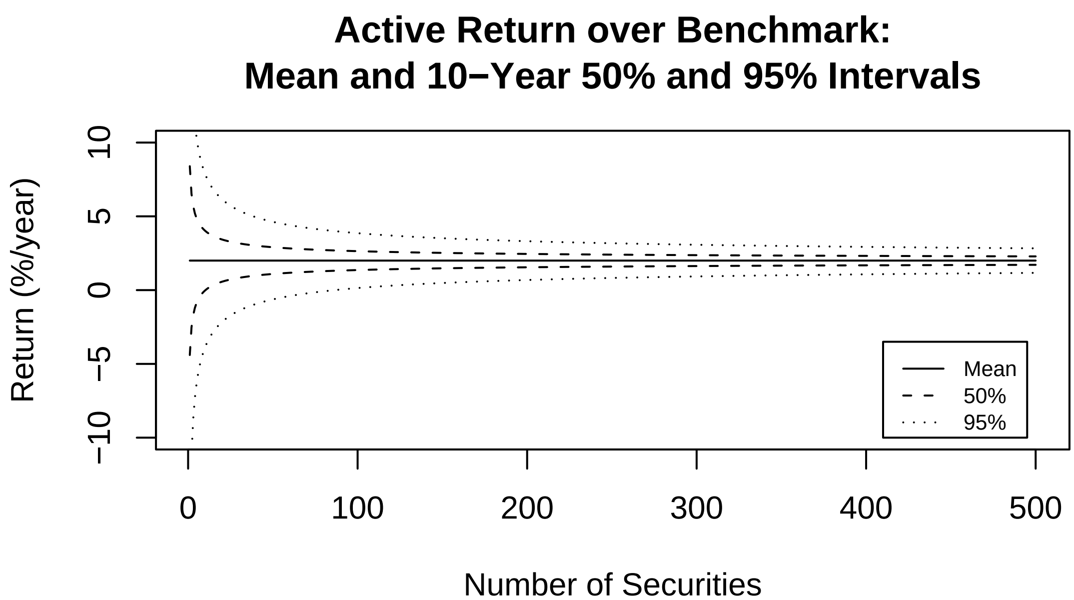

Figure 5: As we add securities, the active portfolio's deviation from the benchmark converges slowly to $+2\%$.

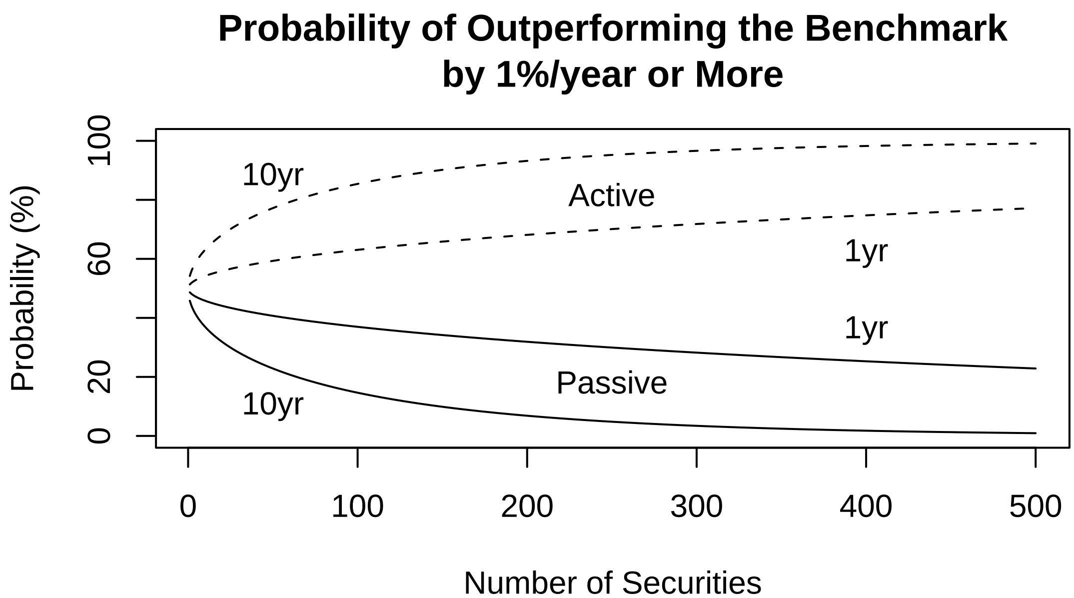

Figure 6: It becomes easier and easier to distinguish the active and passive portfolios as securities are added.

number of securities in the portfolio in both cases. For each number of securities on the horizontal axis, the performance will be between the dashed lines 50% of the time and between the dotted lines 95% of the time. As securities are added, outperformance of the passive portfolio converges slowly to zero, while the the active portfolio converges slowly to 2% above the benchmark. The convergence is slower than in Figure 1 because adding a small amount of idiosyncratic risk does not have much affect on the volatility of the benchmark.[9] These figures serve as a reminder of the difficulty of measuring performance. Although we have given our informed manager a generous 2%/year in superior performance on average, it takes a very-very-well-diversified portfolio before that performance is consistently of larger magnitude than the noise in most years. Nonetheless, this figure does illustrate that it is much easier to measure performance in a large well-diversified portfolio than in a small portfolio.

Figures 4 and 5 show that identifying performance is easier if we are given a 10-year track record. Of course, this does not help much if we have to make decisions about allocations each year. This is why evaluation of managers almost necessarily considers informations beside outperformance of the benchmark.

Summary Traditional analysis of the value of diversification does not apply in the same way to active managers with ability that permits them to outperform the market. In our example, the volatility of the active manager approaches the benchmark volatility as assets are added, but the mean excess performance converges to a positive constant. Hiring a well-diversified active manager might have several advantages. Performance of a well-diversified manager is relatively easy to measure, and such a manager would also do less damage than an undiversified manager if it turns out the manager does not have ability (or no longer has ability).

[9] Analytically, this is because variances of independent random variables add, implying that standard deviations are less than additive.

References

Dybvig, Philip H., and Stephen A. Ross, 1985, "Differential Information and Performance Measurement Using a Security Market Line," *Journal of Finance* **40**, 383–399.

Lintner, John, 1965, "Security Prices, Risk, and Maximal Gains from Diversification," *Journal of Finance* **20**, 587–615.

Markowitz, Harry, 1952, "Portfolio Selection," *Journal of Finance* **7**, 77–91.

Sharpe, William F., 1963, "A Simplified Model for Portfolio Analysis," *Management Science* **2**, 277–293.

Sharpe, William F., 1964, "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk," *Journal of Finance* **19**, 425–442.

Sharpe, William F., 1966, "Security Prices, Risk, and Maximal Gains from Diversification: Reply," *Journal of Finance* **21**, 743–744.

Samuelson, Paul A., 1967, "General Proof that Diversification Pays," *Journal of Financial and Quantitative Analysis* **2**, 1–13.