Perpertual American Put: valuation and verification theorem

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Since this is a valuation problem, it is convenient to work in risk-neutral probabilities. We are solving the following problem, where Γ_t is the indicator for whether the option has been exercised as of time t. We assume the Black-Scholes stock price process (so μ need not be constant but cannot be wild enough to produce arbitrage) with a strike price $X > 0$.

Given S_0 and Γ_{0-},
choose adapted nondecreasing right-continuous $\Gamma_t \in \{0, 1\}$ to
maximize $E \int_{t=0}^{\infty} (X - S_t) e^{-rt} d\Gamma_t$
s.t. $dS_t = rS_t dt + \sigma S_t dZ_t$

The state variables are the stock price S and the exercise indicator Γ and we can write the value function as $V(S, \Gamma)$. Note that $V(S, 1) \equiv 0$ since there are no more cash flows after exercise.

We set up the usual process

$$
\begin{aligned}
M_t &= \int_{s=0}^{\infty} (X - S_s) e^{-rs} d\Gamma_s + e^{-rt} V(S_t, \Gamma_t) \\
e^{rt} dM_t &= (X - S_t) d\Gamma_t - rV(S_t, \Gamma_t) + V_S(S_t, \Gamma_t) dS + \frac{1}{2} V_{SS} S^2 \sigma^2 dt \\
&\quad + V(S_t, \Gamma_t) - V(S_t, \Gamma_{t-}) \\
&= (X - S - V(S, 0)) d\Gamma + (-rV + rSV_S + \frac{1}{2} \sigma^2 S^2 V_{SS}) dt \\
&\quad + SV_S \sigma dZ
\end{aligned}
$$

The process M is a martingale if we have the correct value function and the optimal strategy. From the first term, this implies $V(S, 0) = X - S$ when it is optimal to exercise. When it is not optimal to exercise, the first term is zero (since $V(S, T) > X - S$ and $d\Gamma = 0$) and the last term has mean zero, so we obtain the non-time-dependent version of the Black-Scholes differential equation:

$$-rV + rSV_S + \frac{1}{2} \sigma^2 S^2 V_{SS} = 0.$$

This is a homogeneous Euler equation, and we obtain the characteristic equation by plugging in $V = S^\gamma$ and its derivatives:

$$-r + r\gamma + \frac{1}{2}\sigma^2\gamma(\gamma - 1) = 0$$

Solving for γ (e.g. using the quadratic formula), we find $\gamma = -2r/\sigma^2$ or 1. Therefore, in the no-exercise region we have

$$V(S,0) = \alpha S^{-2r/\sigma^2} + \beta S$$

(Recall that $V(S,1) \equiv 0$, i.e. both coefficients are zero when $\Gamma = 1$.) Now, the put value $V(S,0)$ is always in the range $[0, X]$ and the first term is uniformly bounded by α, which implies $\beta = 0$.

Assume plausibly (and as will be proven in the verification theorem) that there exists a critical stock price $\underline{S}$ such that we exercise the option at any stock price $S \leq \underline{S}$. Then we have $V(S) = X - S$ for $S \leq \underline{S}$. Matching the value at the boundary $\underline{S}$, we have $X - \underline{S} = \alpha \underline{S}^{-2r/\sigma^2}$, which implies

$$\alpha = X\underline{S}^{2r/\sigma^2} - \underline{S}^{1+2r/\sigma^2}.$$

Now, this is the value function above $\underline{S}$ whether or not $\underline{S}$ is optimal. We can compute the optimal value of $\underline{S}$ by maximizing the value function at some large value of S, which is equivalent to maximizing α. Differentiating α with respect to $\underline{S}$ and setting the derivative equal to zero, we have that the optimal boundary is

$$\underline{S} = X\frac{r}{r + \sigma^2/2}.$$

An alternative (and more general) way of solving for $\underline{S}$ is to use the smooth-pasting conditions that the value function and its derivative have to be continouous at $\underline{S}$,

At this point, we have derived the solution and the value function, but we haven't really proven anything yet, since our derivation assumed the form of the optimal strategy, some smoothness, and that the first-order solution for optimal $\underline{S}$ is the actual solution. Here are the claimed strategy and value function:

Strategy: If $\Gamma_{t-} = 0$, choose

$$d\Gamma_t = \begin{cases} 1 & S \leq \underline{S} \\ 0 & \underline{S} < S \end{cases}$$

where $\underline{S} = Xr/(r + \sigma^2/2)$. In other words, exercise immediately if the stock price is at or below the critical level $\underline{S}$. Otherwise, don't exercise and wait until this critical level is reached. Since the stock price is continuous in time, any exercise after time 0 will be at the critical price $\underline{S}$. If $\gamma_{t-} = 1$, choose $d\Gamma_t = 0$ (the only feasible choice).

Value function: $V(S, 1) \equiv 0$ and

$$V(S,0) = \begin{cases} X - S & S \leq \underline{S} \\ \underline{\alpha} S^{-2r/\sigma^2} & \underline{S} < S \end{cases}$$

where

$$\underline{\alpha} = \left(\frac{r}{r + \sigma^2/2} \right)^{2r/\sigma^2} \frac{\sigma^2/2}{r + \sigma^2/2} \, X^{1+2r/\sigma^2}$$

comes from substituting the derived optimal $\underline{S}$ into the expression for α as a function of $\underline{S}$.

Verification Theorem: The claimed solution above is the actual optimal solution for the problem we stated at the outset.

Proof: Step (1) Define M_t using the claimed optimal strategy and the claimed value function. Then, the Bellman equation ensures $E[dM_t] = 0$ so M_t is a local martingale. Furthermore, it is easy to see $0 \leq M_t \leq X$. Therefore, M is a martingale because it is a bounded local martingale.

Step (2) Define M_t using the claimed value function and any feasible strategy. Since the value function satisfies $0 \leq V(S, \Gamma) \leq X$ everywhere, and $\lim_{T\uparrow\infty} e^{-rT} = 0$, the transversality condition $\lim_{T\uparrow\infty} E[e^{-rT} V(S_T, \Gamma_T)] = 0$.

Step (3) We want to show that M is a local supermartingale under all strategies. First note that if $\Gamma_{t-} = 1$, then the only strategy is to choose $d\Gamma_t = 0$ and trivially $dM = 0$. If $\Gamma_{t-} = 0$, then we have two cases.

If $S_t > \underline{S}$, $d\Gamma = 0$ (no exercise, the claimed optimal strategy) implies $E[dM_t] = 0$ (by construction). Furthermore, $d\Gamma = 1$ implies that $E[dM] = -V(S_t, 0) + X - S_t < 0$, as can be proven by noting (as can be verified by simple calculations) that $V(\underline{S}, 0) = X - \underline{S}$, $V_S(\underline{S}, 0) = -1$, and for $S > \underline{S}$, $V_{SS}(S, 0) > 0$. Therefore, $-V(S_t, 0) + X - S_t < 0$ is equivalent to $V(S_t, 0) > (X - \underline{S}) - (S - \underline{S})$ can be interpreted as the support condition for the convex function $V(S_t, 0)$ at $\underline{S}$, since at $\underline{S}$, V has value $x - \underline{S}$ and slope -1.

If $S \leq \underline{S}$, $d\Gamma = 1$ (exercise, the claimed optimal strategy), which imples $E[dM_t] = -V(S,0) + X - V = 0$. Furthermore, $d\Gamma = 0$ implies (by Itô's lemma) that $E[dM] = -rVdt + V_S E[dS] + (1/2)\sigma^2 S^2 V_{SS} dt$. However, in this region, $V(S,0) = X - S$ so $V_S(S,0) = -1$ and $V_{SS} S, 0 = 0$. Therefore, $E[dM] = -r(X-S)dt - rSdt = -rXdt < 0$.

In all cases, $E[dM] \leq 0$ implying that M is a local supermartingale.

Step (4) Consider M as in step (3) for the claimed value function and any feasible strategy. Without loss of generality, we can restrict attention to strategies that only exercise options at or in the money ($(X - S_t)d\Gamma_t \geq 0$), since strategies that exercise out of the money ($(X - S_t)d\Gamma_t < 0$) with positive probability are dominated by strategies that never exercise in those states. For the remaining strategies that only exercise in the money, we have that $0 \leq M_t \leq X$ uniformly, and therefore M must be a supermartingale because it is a bounded local supermartingale.

Step (5) First consider M defined using the claimed value function and the claimed optimal strategy. By (1), we can write

$$M_0 = \lim_{T \uparrow \infty} E[\int_{s=0}^{T} (X - S_s) e^{-rs} d\Gamma_S] + \lim_{T \uparrow \infty} E[e^{-rT} V(S_T, \Gamma_T)],$$

provided the limits exist. The limits do exist by the transversality condition (Step (2)), which also implies the second term is 0. Also, the first term is the objective function of our problem, which confirms $M_0 = V(S_0, \Gamma_0)$ is indeed the initial value function of our problem given the claimed optimal strategy.

Now consider M defined using the claimed value function and any arbitrary feasible strategy not already eliminated in Step (4). By (4), we can write

$$M_0 \geq \lim_{T \uparrow \infty} E[\int_{s=0}^{T} (X - S_s) e^{-rs} d\Gamma_S] + \lim_{T \uparrow \infty} E[e^{-rT} V(S_T, \Gamma_T)],$$

provided the limits exist. The limits do exist: the first limit exists because the expression is nondecreasing in T and bounded above by X, while the second limit equals 0 by the transversality condition in Step (2). Since $M_0 = V(S_0, \Gamma_0)$ and the first limit is the value of the arbitrary strategy, we have that the claimed optimum is at least as high as the value of any feasible strategy. Since the claimed optimum is feasible, it is also optimal. ∎