

Employee Reload Options: Pricing, Hedging, and Optimal Exercise

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Reload options, call options granting new options on exercise, are popularly used in compensation. Although the compound option feature may seem complicated, there is a distribution-free dominant policy of exercising reload options whenever they are in the money. The optimal policy implies general formulas for numerical valuation. Simpler formulas for valuation and hedging follow from Black–Scholes assumptions with or without continuous dividends. Time vesting affects the optimal policy, but numerical results indicate that it is nearly optimal to exercise in the money whenever feasible. The results suggest that reload options produce similar incentives as employee stock options and share grants.

The valuation of options in compensation schemes is important for several reasons. Valuations are needed for preparing accounting statements and tax returns, and more generally for understanding what value has been promised to the employees and what residual value remains with the shareholders. Furthermore, understanding the hedge ratios and the overall shape of the valuation function clarifies the employee's risk exposure and incentives. This article studies the optimal exercise and valuation of a relatively new but increasingly commonplace type of employee stock option, the reload option. These options have attracted a fair amount of controversy; we believe that in large part this controversy is due to a seemingly complex structure. In the Statement of Financial Accounting Standards (SFAS) 123 [Financial Accounting Standards Board (1995)], paragraph 186 concludes, "The Board continues to believe that, ideally, the value of a reload option should be estimated on the grant date, taking into account all of its features. However, at this time, it is not feasible to do so." On close examination, however, these options are in fact comparatively simple to analyze and understand. Moreover, under Black and Scholes (1973) assumptions on the stock price, we are able to provide explicit valuation and hedging formulas. We hope our analysis helps to demystify reload options and permit a more focused debate.

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Reload options, sometimes referred to as restoration or replacement options, have been an increasingly common form of compensation for executives and other employees: 17% of new stock option plans in 1997 included some type of reload provision, up from 14% in 1996 [see Reingold and Spiro (1998)]. Because of this increased popularity, reload plans have received increased scrutiny and have often been met with skepticism. According to one study at Frederic W. Cook and Company (1998), “at least one major institutional investor considers the presence of this feature in a plan to be grounds for a ‘no’ vote.” Others argue that reloads have positive benefits such as encouraging stock ownership. Our analysis confirms the warning that each reload option is probably worth significantly more than a single traditional option, but otherwise debunks many of the sensational criticisms of reload options.

The reload option has the feature that if the option is exercised prior to maturity and the exercise price is paid with previously owned shares, the holder is entitled to one new share for each option exercised plus new options which reload or replace some of the original options. Like most contracts that are not standardized by regulation of an exchange or government, there is substantial variation in the terms offered in practice. Frederic W. Cook and Company (1998) and Hemmer, Matsunaga, and Shevlin (1998) describe some of the common variants and their frequencies. Based on the summary statistics in these two articles,¹ the most common plans seem to be our leading case, which allows unlimited reloads without any period of time for vesting, and our other case with time vesting, which allows unlimited reloads subject to a waiting period (most commonly 6 months) between reloads. Another variation in the contracts is in the number of options granted on exercise. For example, some plans issue reload options for shares tendered to cover withholding tax on top of shares tendered to cover the strike price and some plans issue reload options which replace all the options exercised. We choose, however, to focus on the more common case in which one new option is issued for each share tendered to pay the exercise price. This case is consistent with the definition of a reload option given in SFAS no. 123, paragraph 182 (1995). However, we should warn the reader that our results do not necessarily apply to more exotic reload options.

For most of the article we assume frictionless markets, however, it is important to note that our main result on optimal exercise, Theorem 1, relies only on simple dominance arguments. The main assumptions we use to derive the optimal exercise policy are (1) the employee is permitted to retain new

¹ Hemmer, Matsunaga, and Shevlin (1998) analyzed a sample of 246 firms with reload options in their compensation plans. Of these, 27 plans had extensive restrictions on multiple reloads, usually to a single reload. Of the remaining 219 plans, 53 had explicit vesting requirements for the options. Frederic W. Cook and Co. (1998) analyzed 40 plans with reload options. Of these, 10 had some restriction on performance vesting. Thirteen plans restricted severely the number of reloads (again usually to one), while the remainder permitted an unlimited number of reloads, “often” with a six-month vesting period.

shares of stock from the exercise, (2) the employee either owns or can borrow enough shares to pay the exercise price, (3) the stock price and other components of the employee's compensation are unaffected by the exercise decision, and (4) there are no taxes or transaction costs. Under these assumptions, the optimal exercise policy is to exercise whenever the option is in the money and this policy is quite robust to restrictions on the employee's ability to transact in the stock. As a result, we provide explicit market values of the reload option in Theorem 2. Thus, under our assumptions, we can provide accurate descriptions of how much value the firm has given up and how the firm can hedge its exposure without needing to model explicitly the employee's preferences or other components of the employee's portfolio. This is true even if the market value of the options is different from the private value to the employee.

Our interest in reload options derives from Hemmer, Matsunaga, and Shevlin (1998), who documented the use of the various forms of the reload option in practice, demonstrated the optimal exercise policy, and valued the reload option using a binomial model for the stock price and a constant interest rate. Arnason and Jagannathan (1994) employ a binomial model to value a reload option that can be reloaded only once. Saly, Jagannathan, and Huddart (1999) value reload features under restrictions on the number of times the employee can exercise in a binomial framework. Our contribution is to provide values for the reload option for more general stochastic processes governing the interest rate, dividends, and stock price, under the assumption that there is no arbitrage in complete financial markets. This is important since (1) our result does not rely on choosing a binomial approximation under which to evaluate the option, and (2) our approach yields simple valuation and hedging formulas which can be computed easily in terms of the maximum of the log of the stock price. We also examine the impact of time vesting requirements on the optimal exercise policy and valuation. Our analysis suggests that time vesting has a relatively small impact on valuation but may dramatically affect the optimal exercise policy.

Our results shed light on some of the controversy about reload options. Some sensational claims about how bad reload options are have appeared in the press [see, e.g., Reingold and Spiro (1998) and Gay (1999)]. For example, there is a suggestion that being able to exercise again and again and get new options represents some kind of money pump, or that this means that the company is no longer in control of the number of shares issued. However, even with an infinite horizon (which can only increase value compared to a finite horizon), the value of the reload option lies between the value of an American call and the stock price. Furthermore, given that the exercise price is paid in shares, the net number of new shares issued under the whole series of exercises is bounded by the initial number of reload options just as for ordinary call options. Another suggestion in the press is that the reload options might create bad incentives for risk taking or for reducing

dividends. In general, it is difficult to discuss incentives without information on other pieces of an employee's compensation package and knowledge of what new pieces will be added and in what contingencies. However, to a first approximation, the valuation and the hedge ratio, or delta, characterize an option's contribution to an employee's incentives. Indeed, we will see that the replicating portfolio holds between zero and one share of the stock and the delta per unit value is quite similar to that of a European call option. Thus it appears that incentives from reload options are not so different from the corresponding incentives for traditional employee stock options.

1. Background and Discussion

Reload options were first developed in 1987 by Frederic W. Cook and Company for Norwest Corporation and were included in 17% of new stock option plans in 1997, up from 14% in 1996 [Reingold and Spiro (1998) and Gay (1999)]. Reload options are essentially American call options with an additional bonus for the holder. When exercising a reload option with a strike price of K when the stock price is S , the holder receives one share of stock in exchange for K . In addition, when the strike price is paid using shares valued at the current market price (K/S shares per option), the holder also receives for each share tendered a new reload option of the same maturity, but with a strike equal to the stock price at the time of tender. For example, if an employee owns 100 reload options with a strike of \$100 and the stock price at time of exercise is \$125, 80 shares of stock with a total market value of $\$125 \times 80 = \$10,000$ are required to pay the strike price of $\$10,000 = \100 per option \times 100 options. Assuming frictionless buying and selling or at least pre-existence of shares needed to tender in the employee's portfolio, the exercise will net 20 ($= 100 - 80$) shares of stock with market value of \$2,500 ($= \125 per share \times 20 shares), and in addition the employee will receive 80 new reload options (one for each share tendered), each having a strike price of \$125 and the same maturity as the original reload options. As for other types of options issued to employees, there is some variation in reload option contracts used in practice. For example, a small proportion [about 10% according to Hemmer, Matsunaga, and Shevlin (1998)] of the options allow only a single reload, so the new options are simple call options. We analyze the more common case in which many reloads are possible. Another variation in practice is that each new option may require a vesting period before it can be exercised. We focus primarily on the simpler case in which the option can be exercised anytime after issue, but we analyze the case with vesting in Section 6. The analysis there includes numerical analysis in a trinomial model and useful bounds on the value in continuous time. It is of interest that the value under the optimal exercise policy is not much different from the value of exercising whenever the option is in the money at multiples of the vesting period.

Before proceeding to the analytic valuation of a reload option, it is useful to establish no-arbitrage upper and lower bounds on the option price. Besides developing our intuition, these bounds will help us to assess claims we have seen in the press that suggest that there is no limit to the value of a reload option that can be reloaded again and again, especially if (as is sometimes the case) the new options issued have a life extending beyond the life of the one previously exercised; these options would have a value less than our upper bound for the reload option value that is applicable even if the option allows an unlimited number of reloads and has infinite time to maturity.

The useful lower bound on a reload option's value is the value of an American call option. The reload option can be worth no less because the holder can obtain the American call's payoff by following the American call's optimal exercise strategy without ever exercising the reloaded options.

The upper bound on a reload option is the underlying stock price, no matter how many reloads are possible and no matter how long the maturity of the option, even if it is infinitely lived. This observation debunks effectively the popular claim that not having a limit on the number of reloads or the overall maturity means that the company is losing control of how many options or shares can be generated. To demonstrate this upper bound requires a bit of analysis. Arguing along the lines of the example above, the first exercise (say at price S_1) yields the employee, for each reload option, $(1 - K/S_1)$ shares and K/S_1 new reload options with strike S_1 . At the second exercise (say at price S_2), the employee nets an additional $(K/S_1)(1 - S_1/S_2)$ shares, for a total of $(1 - K/S_1) + (K/S_1)(1 - S_1/S_2) = (1 - K/S_2)$ shares from both exercises, and $(K/S_1)(S_1/S_2) = K/S_2$ new options with strike S_2 . After the i th exercise, the employee will have in total $(1 - K/S_i)$ shares and K/S_i new options with strike S_i . Therefore, no matter how far the stock price rises, the employee will always have less than one share per initial reload option, and the value is further reduced because the employee will not receive the early dividends on all of the shares. Therefore the employee would be better off holding one share of stock and getting the dividends for all time, and therefore the stock price is an upper bound for the value of a reload option.

Any discussion of incentive effects of employee stock options is somewhat speculative, given that the employee's valuation may differ from the market valuation. Nonetheless we can get a feel for the incentive effects associated with reload options by previewing some results generated later in the article under standard Black-Scholes assumptions. Figures 1 and 2 compare the reload option values and hedge ratios to those for a European call option as a function of moneyness. These figures confirm that the reload option value is less than the stock price and the hedge ratio is less than or equal to one. While the reload option is generally more valuable than the European call option, the general incentive effects are quite similar. One measure of the incentive effects of option compensation is the delta per unit value of the option. Figure 3 plots this measure versus moneyness for reload and

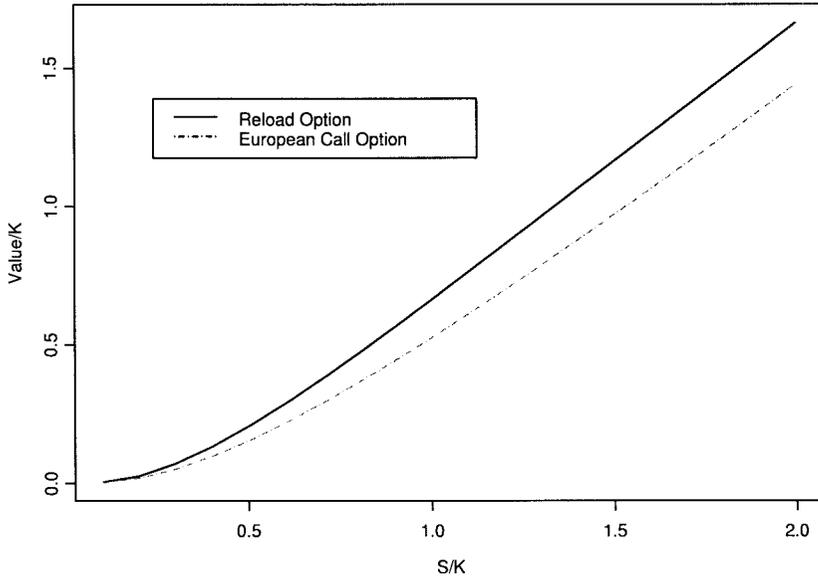


Figure 1
Comparison of reload option value with European call option value

This shows the value of a reload option (upper curve) and a European call option (lower curve) with 10 years to maturity as a function of S/K , assuming an annual interest rate of 5%, no dividends, and volatility of 30%.

European call options. In these terms, we see that the incentive effects of the reload option are quite similar to those of a European call option with the same market value.

Before proceeding to the formal analysis, it is worthwhile noting some simple comparative statics. First, the value of a reload option, like the value of a call option, is decreasing in the strike price. It is increasing in the stock price for cases in which changing the stock price is a simple rescaling of the process. As in the case of the American call option, the value of the reload option is increasing in time to maturity. Given the value of the underlying investment, a higher dividend rate decreases the value of a reload, since what you get from each exercise is less. Finally, we would normally expect the value of a reload option to increase with volatility and the risk-free rate; we show that under Black and Scholes (1973) assumptions, this is the case.²

2. Underlying Stock Returns and Valuation

Our model has two primitive assets, a locally riskless asset, “the bond,” with price process $B(t) > 0$, and a risky asset, “the stock,” with price process $S(t) > 0$. Time t takes values $0 \leq t \leq T$ and all random variables and random

² These conclusions cannot be completely general for the same reasons put forward by Jagannathan (1984).

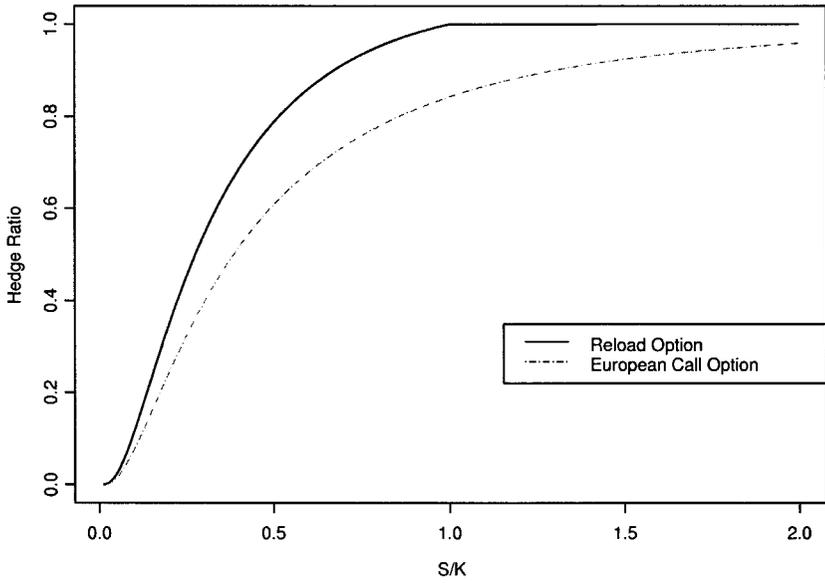


Figure 2

Comparison of reload option hedge ratio with European call option hedge ratio

This shows the hedge ratio for a reload option (upper curve) and European call (lower curve) with 10 years to maturity as a function of S/K , assuming an annual interest rate of 5%, no dividends, and volatility of 30%.

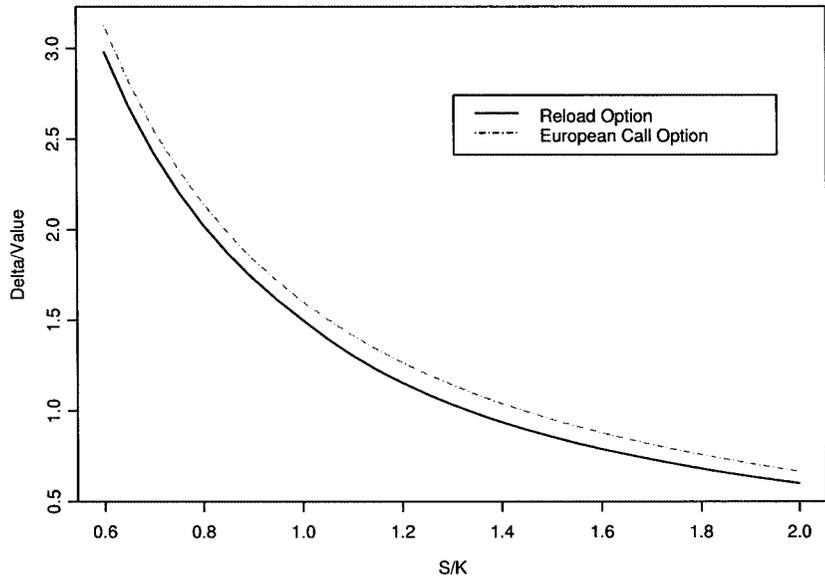


Figure 3

Comparison of reload option delta per unit value with that for a European call option

This shows the delta per unit value for a reload option (lower curve) and European call (upper curve) with 10 years to maturity as a function of S/K , assuming an annual interest rate of 5%, no dividends, and volatility of 30%.

processes are defined on a common filtered probability space.³ We assume that $S(t)$ is a special semimartingale that is right continuous and left limiting. The risky asset may pay dividends, and the nondecreasing right-continuous process $D(t) > 0$ denotes the cumulative dividend per share. We actually require very little structure on the bond price process $B(t)$; positivity and measurability is enough for most of our results, and finite variation is needed for another. Of course, we would normally expect much more structure on $B(t)$; if interest rates exist and are positive, then $B(t)$ is increasing and differentiable. For some particular valuation results (but not the proof of the optimal strategy), we will assume that $S(t)$ can only jump downward (as it would on an ex dividend date), but not upward. These particular valuation results will be used to obtain a simple formula for the Black–Scholes case with or without continuous dividends.

To value a cash flow, it is equivalent to use a replicating strategy or risk-neutral valuation. Consider first how a replicating strategy would work. Suppose we want to replicate a payoff stream whose cumulative cash flow is given by the nondecreasing right-continuous process $C(t)$. [Taking as primitive the cumulative cash flow $C(t)$ admits lumpy withdrawals as well as continuous withdrawals. For example, choosing $C(t) = 0$ for $t < T$ and $C(T) > 0$ would correspond to a single withdrawal at the end.] To account for possible cash flows at time 0, and more generally to allow for values of a random process before and after any time t , we will use the values $0-$ or $t-$, respectively, to indicate what is true just before these times. Our usage is also consistent with using this notation for the left limit whenever the left limit is defined. For example, $C(t) - C(t-)$ denotes the amount of cash flow at time t , whether $t > 0$ or $t = 0$. A replicating strategy is defined by two predictable processes, the number of bonds held $\alpha(t)$ and the number of shares held $\theta(t)$. The wealth process

$$W(t) = \alpha(t)B(t) + \theta(t)S(t) \tag{1}$$

is constrained to be nonnegative and evolves according to

$$dW(t) = \alpha(t) dB(t) + \theta(t) dS(t) + \theta(t) dD(t) - dC(t). \tag{2}$$

Stating matters this way does not rule out suicidal strategies (such as a doubling strategy run in reverse), but such strategies are not relevant once we define the value of a cumulative cash flow $C(t)$ as the smallest value of $W(0-)$ in a consistent replicating strategy.

To rule out arbitrage, we could make assumptions about the underlying stock and bond processes, but instead we will simply assume the existence

³ If the space is $(\Omega, \mathcal{F}, P, \{\mathcal{F}(t)\}_{t \in [0, T]})$, we denote by $E_t[\cdot]$ expectation conditional on $\mathcal{F}(t)$. All random processes are measurable with respect to this filtration. We will also consider expectations under the “risk-neutral” probability measure P^* , with E_t^* defined analogously to E_t . See Karatzas and Shreve (1991) for definitions of these terms.

of a risk-neutral probability measure P^* , equivalent to P (meaning that P and P^* agree on what events have positive probability), which can be used to price all assets in the economy. Under P^* , investing in the stock is a fair gamble in present values, and we have that for $s \geq t$

$$\frac{S(t)}{B(t)} = E_t^* \left[\frac{S(s)}{B(s)} + \int_t^s \frac{1}{B(u)} dD(u) \right]. \quad (3)$$

We will assume complete markets, which implies P^* is unique and, moreover, it is well known that in this circumstance we can write the time 0– price of any consumption withdrawal stream as

$$E^* \left[\int_{t=0-}^T \frac{1}{B(t)} dC(t) \right]. \quad (4)$$

This expression is equal to $W(0-)$ in any efficient candidate replicating strategy. It is less than $W(0-)$ for a wasteful strategy that throws away money. Money could be thrown away by never withdrawing it ($W(T) > 0$) or by following a suicidal policy. The valuation in Equation (4) is the relevant one, since we are not interested in wasteful strategies.

3. Reload Options With Discrete Exercise

The reload option, with strike price K and expiration date T , is an option which, if exercised on or before the expiration date and the exercise price is paid with previously owned shares, entitles the holder to one share for each option exercised plus one new reload option per share tendered. The new reload option has a strike price equal to the current stock price and it has the same expiration date as the original option. Our basic assumption for this section is that the employee is initially holding enough shares to pay the exercise price (or at least the necessary shares can be borrowed) and it is feasible to retain the shares upon exercise. If the employee does exercise and retain the new shares, we see that the payoff to exercising a single reload option with strike price K at time $t \leq T$ is $1 - K/S(t)$ shares plus $K/S(t)$ new reload options with strike price $S(t)$ and expiration date T . Of course, the employee must decide when to exercise these new options.

There is a slight technical issue concerning the definition of payoffs given the possibility of continuous exercise of reload options. To finesse this issue, we consider in this section exercise at a discrete grid of dates. The following section will consider the continuous case, for which there is a singular control that can be handled very simply by looking at well-defined limits of the discrete case. [This is analogous to the singular control of regulated Brownian motion, as in Harrison (1985).]

For the rest of this section we assume that exercise is available only on the set of nonstochastic times $\{t_1, t_2, \dots, t_n\}$, where $0 = t_1 < t_2 < \dots < t_n = T$.

An exercise policy is defined to be an increasing family of stopping times, τ_i taking values on the grid with $t_1 \leq \tau_1 < \dots < \tau_i < \dots$. For the derivation of the optimal strategy, we will assume the following:

Assumption 1. *The employee is always free to hold additional shares.*

Assumption 2. *The employee is always holding enough shares to pay the exercise price (or at least can borrow the necessary shares).*

Assumption 3. *The exercise decision itself does not affect the employee's compensation, the stock price, or dividend payments, for example, through the dependence of future wages on exercise, through a dilution of shares, or through signaling.*

Assumption 4. *The dividend payments are nonnegative and the stock price is strictly positive. The employee prefers more consumption to less and can eventually convert dividend payments and share receipts into desirable subsequent consumption.*

Assumption 5. *There are no taxes or transaction costs.*

Most of these assumptions are quite weak and allow for the possibility that the employee may face restrictions on the selling of shares of the stock. The assumption of no taxes is a strong assumption, but without this assumption we cannot say much. For example, it may be optimal to defer exercise into a new tax year to delay recording of income. We think it is plausible that this will not affect the market value by very much, but this remains to be proven.

We first provide an analysis of the payoffs from multiple exercise decisions. The number of shares received after the first exercise is $(1 - K/S(\tau_1))$ and the employee receives $K/S(\tau_1)$ new reload options with strike price $S(\tau_1)$. The number of shares received after the exercise of the new reload options is $(K/S(\tau_1) - K/S(\tau_2))$. So the cumulative number of shares received after the second exercise is $(1 - K/S(\tau_1)) + (K/S(\tau_1) - K/S(\tau_2)) = (1 - K/S(\tau_2))$ and the employee also holds $K/S(\tau_2)$ new reload options. In general, after the i th exercise, the employee will have received $(1 - K/S(\tau_i))$ cumulative shares and will hold $K/S(\tau_i)$ new reload options with strike price $S(\tau_i)$, where we use the convention $S(\tau_0) = K$. (This is the same as the result derived in Section 1, only now in formal notation.) At a general time t , the employee has received $1 - \frac{K}{X(t)}$ cumulative shares and holds $\frac{K}{X(t)}$ reload options with strike price $X(t)$, where $X(\cdot)$ is the strike or exercise price process defined by

$$X(t) = \begin{cases} K & 0 \leq t < \tau_1 \\ S(\tau_1) & \tau_1 \leq t < \tau_2 \\ S(\tau_2) & \tau_2 \leq t < \tau_3 \\ \vdots & \end{cases}, \tag{5}$$

since the strike price is initially K and later is the price of the most recent exercise.

While at first glance a problem with multiple exercise decisions may appear difficult, the derivation of the optimal exercise policy is straightforward. Notice that the actual position of the employee is the sum of any endowment or inheritance, compensation including the position from the reload exercise strategy above, a net trade reflecting purchases and sales of the stock, and any other investments. An employee who can always hold more shares will prefer to receive shares earlier (to collect dividends) and will prefer to obtain more shares rather than fewer shares. Fortunately the strategy of exercising whenever the reload options are in the money gives the employee more shares earlier than any other strategy.

Theorem 1. *It is an optimal policy to exercise the reload option whenever it is in the money, and refrain from exercising whenever it is out of the money. This strategy results in the exercise process $X^*(t)$, where*

$$X^*(t) = M^n(t) \equiv \max\{K, \max\{S(t_i) | t_i \leq t\}\} \quad (6)$$

is the nondecreasing process that describes the strike price as a function of time under this optimal strategy on the grid with n points. This is the only optimal strategy (up to indifference about exercising at dates when the option is at the money) if the stock price can always fall between grid dates (which we think of as the ordinary case).

Proof. Without loss of generality, assume that there is no exercise when the options are at the money (this is irrelevant for payoffs). First we show that $X^*(t)$ is as claimed if we exercise at exactly those grid dates when the option is in the money. When $t < \tau_1$, no exercise has taken place and the maximum in the definition must be K (or there would have been exercise at the first date greater than K , contradicting $t < \tau_1$). When $\tau_1 < t$, there has been at least one exercise. In this case there must have been an exercise at the first date achieving the largest price so far (which is necessarily larger than K or there would have been no exercise so far). And there cannot have been any subsequent exercise, since the option has not been in the money since then. This shows that $M^n(t)$ is indeed the exercise price at t .

Now, we need to show that this is an optimal strategy for the employee. Fix any feasible exercise policy $X(t)$ along with associated managerial, consumption, and portfolio choice decisions and let $\theta(t)$ be the process representing the number of shares of the stock held at time t . Consider switching from $X(t)$ to our candidate optimum $X^*(t)$, holding all other decisions fixed

outside of the exercise decision. Notice $X^*(t) \geq X(t)$. The process $\theta^*(t)$ which describes the number of shares held at time t is given by

$$\begin{aligned} \theta^*(t) &= \theta(t) - \left(1 - \frac{K}{X(t)}\right) + \left(1 - \frac{K}{X^*(t)}\right) \\ &= \theta(t) + \frac{K}{X(t)} - \frac{K}{X^*(t)} \\ &\geq \theta(t). \end{aligned}$$

The switch is feasible since, by Assumption 1, the employee can always increase the holding of shares, and by Assumption 2, the employee always has enough shares to pay the exercise price. The shift in exercise strategy does not affect the dividend payments, stock price, outside consumption, or portfolio payoffs by Assumption 3 and results in an additional cumulative dividend payment of $\int_0^T \left(\frac{K}{X(t)} - \frac{K}{X^*(t)}\right) dD(t) \geq 0$ and $\frac{K}{X(T)} - \frac{K}{X^*(T)} \geq 0$ additional shares at the expiration of the option. Since the employee prefers more to less, the stock price is strictly positive, and dividends are nonnegative by Assumption 4, the X^* strategy is at least as good as $X(t)$ and is strictly preferred if $X(T) \neq X^*(T)$, since by Assumption 4 the employee can eventually convert extra shares into desirable consumption.

If the stock price can always decrease between grid dates, then this optimal strategy is unique; any other strategy would have a positive probability of missing the maximal stock price on grid dates if we do not exercise, and then the term corresponding to shares at T will be smaller than under the optimum. ■

Theorem 1 admits the possibility that there are optimal strategies in which we do not exercise whenever the option is in the money, but only for the esoteric case in which it is known in advance the stock price will rise for certain between discrete dates.⁴ This esoteric case is not consistent with what we know about actual stock prices, and we think of it as a mathematical curiosity. Therefore we should think of the policy of exercising when the option is in the money as optimal.

To study the optimal exercise strategy, it is useful to view the proceeds of exercise as the net receipt of shares. Recall from our previous analysis, by following an arbitrary exercise policy, that the employee will receive $(1 - K/X(t))$ cumulative shares at time t . However, the ultimate disposition of the shares received should have no effect on the market value since any net trade has zero market value (although this may not be the case for the private value to the employee). Assuming these shares will be held until the

⁴ This does not necessarily imply arbitrage if, for example, the stock return in the period will be either half or twice the risk-free rate.

maturity of the option, this results in a market value of an arbitrary exercise policy X as

$$E^* \left[\frac{S(T)}{B(T)} \left(1 - \frac{K}{X(T)} \right) + \int_0^T \left(1 - \frac{K}{X(t)} \right) \frac{1}{B(t)} dD(t) \right]. \quad (7)$$

We emphasize that Equation (7) is the market value of an exercise policy, not the private value to the employee. Under the optimal exercise policy, the market value is given by setting $X(t) = X^*(t)$, which results in the market value

$$E^* \left[\frac{S(T)}{B(T)} \left(1 - \frac{K}{M^n(T)} \right) + \int_0^T \frac{1}{B(t)} \left(1 - \frac{K}{M^n(t)} \right) dD(t) \right]. \quad (8)$$

On the other hand, for valuation and hedging, it is more useful to treat each exercise as a cash event. In other words, upon granting shares, the firm values them at the market price. This perspective gives us the alternative valuation formula for an arbitrary exercise policy $X(t)$,

$$E^* \left[\sum_{i|\tau_i \leq T} \frac{1}{B(\tau_i)} \frac{K}{X(\tau_i-)} (S(\tau_i) - X(\tau_i-)) \right]. \quad (9)$$

Of course, Equations (9) and (7) should have the same value for a given exercise policy. This is the next result.

Lemma 1. *Given any exercise policy, we have that Equations (9) and (7) are the same.*

Proof. From simple algebra and the definition of $X(t)$ [recall $X(\tau_i-) = S(\tau_{i-1})$ and $S(\tau_0) \equiv K$],

$$\begin{aligned} & E^* \left[\sum_{i|\tau_i \leq T} \frac{1}{B(\tau_i)} \frac{K}{X(\tau_i-)} (S(\tau_i) - X(\tau_i-)) \right] \\ &= E^* \left[\sum_{i|\tau_i \leq T} \frac{S(\tau_i)}{B(\tau_i)} \left(\frac{K}{X(\tau_i-)} - \frac{K}{S(\tau_i)} \right) \right] \\ &= E^* \left[\sum_{i|\tau_i \leq T} \frac{S(\tau_i)}{B(\tau_i)} \left(\frac{K}{S(\tau_{i-1})} - \frac{K}{S(\tau_i)} \right) \right]. \end{aligned}$$

Doob's optional sampling theorem and Karatzas and Shreve's (1991) problem 1.2.17 allow us to write Equation (3) for the stopping time τ_i on the event $\tau_i \leq T$, so we have

$$\begin{aligned} &= E^* \left[\sum_{i|\tau_i \leq T} E_{\tau_i}^* \left[\frac{S(T)}{B(T)} + \int_{\tau_i}^T \frac{1}{B(t)} dD(t) \right] \left(\frac{K}{S(\tau_{i-1})} \frac{K}{S(\tau_i)} \right) \right] \\ &= E^* \left[\sum_{i|\tau_i \leq T} \left(\frac{S(T)}{B(T)} + \int_{\tau_i}^T \frac{1}{B(t)} dD(t) \right) \left(\frac{K}{S(\tau_{i-1})} \frac{K}{S(\tau_i)} \right) \right]. \end{aligned}$$

Let $\gamma = \max\{i|\tau_i \leq T\}$ be the index of the last exercise or 0 if there is no exercise. Obviously $X(T) = S(\tau_\gamma)$. We can then write

$$\sum_{i=1}^{\gamma} \frac{S(T)}{B(T)} \left(\frac{K}{S(\tau_{i-1})} - \frac{K}{S(\tau_i)} \right) = \frac{S(T)}{B(T)} \left(1 - \frac{K}{X(T)} \right)$$

because the sum is telescoping.

It is now useful to define

$$\begin{aligned} a_i &= \int_{\tau_i}^T \frac{1}{B(t)} dD(t) \quad i = 1, \dots, \gamma \\ a_0 &= \int_0^T \frac{1}{B(t)} dD(t) \\ b_i &= \frac{K}{S(\tau_i)} \quad i = 0, \dots, \gamma \end{aligned}$$

and recall the simple identity (summation by parts)

$$\sum_{i=1}^{\gamma} b_{i-1} a_{i-1} - b_i a_i = \sum_{i=1}^{\gamma} a_i (b_{i-1} - b_i) + \sum_{i=1}^{\gamma} b_{i-1} (a_{i-1} - a_i) = a_0 b_0 - a_\gamma b_\gamma$$

which leads to (here we use the convention $\tau_0 \equiv 0$ and $S(\tau_0) \equiv K$)

$$\begin{aligned} &\sum_{i=1}^{\gamma} \left(\int_{\tau_i}^T \frac{1}{B(t)} dD(t) \right) \left(\frac{K}{S(\tau_{i-1})} - \frac{K}{S(\tau_i)} \right) \\ &= \int_0^T \frac{1}{B(t)} dD(t) - \sum_{i=1}^{\gamma} \frac{K}{S(\tau_{i-1})} \int_{\tau_{i-1}}^{\tau_i} \frac{1}{B(t)} dD(t) - \frac{K}{S(\tau_\gamma)} \int_{\tau_\gamma}^T \frac{1}{B(t)} dD(t) \\ &= \int_0^T \left(1 - \frac{K}{X(t)} \right) \frac{1}{B(t)} dD(t) \end{aligned}$$

which completes the proof. ■

As a result, we have the following valuation result:

Theorem 2. *For the optimal exercise policy in Theorem 1, we have that the market value*

$$E^* \left[\frac{S(T)}{B(T)} \left(1 - \frac{K}{M^n(T)} \right) + \int_0^T \frac{1}{B(t)} \left(1 - \frac{K}{M^n(t)} \right) dD(t) \right] \quad (10)$$

can be written equivalently as

$$E^* \left[\sum_{j=1}^n \frac{1}{B(t_j)} \frac{K}{M^n(t_{j-})} (M^n(t_j) - M^n(t_{j-})) \right]. \quad (11)$$

Proof. From Lemma 1, Equations (9) and (7) have the same value. Set $X(t) = M^n(t)$. On date t_j when there is no exercise (i.e., $t_j \neq \tau_i$ for any i), $M^n(t_j) - M^n(t_{j-}) = 0$ and consequently the j th term in Equation (11) is 0. The other dates are exercise dates, and the term in Equation (11) equals the corresponding term in Equation (9). ■

Using Equation (11) in simulations on a fine grid is probably a good way to evaluate reload options for general processes. In view of the dependence on the maximum, using the idea from Beaglehole, Dybvig, and Zhou (1997) of drawing intermediate observations from the known distribution of the maximum of a Brownian bridge should accelerate convergence significantly.

4. Valuation of Reload Options With Continuous Exercise

When the employee can exercise the reload option continuously in time, there is a technical issue of how to define payoffs. If we restrict the employee to exercising only finitely many times, we do not achieve full value, while if the employee can exercise infinitely many times it may not be obvious how to define the payoff. We finesse these technical issues by looking at exercise on a continuous set of times as a suitable limit of exercise on a discrete grid as the grid gets finer and finer. Given the simple form of the optimal exercise policy, this yields formulas in the continuous-time case that are just as simple as the formulas for discrete exercise. We derive these formulas in this section, and we specialize them to the Black–Scholes world in the following section.

Consider first the valuation formula of Equation (8) based on the corresponding discrete optimal strike price process of Equation (6). As the grid becomes finer and finer, the strike price process converges from below to its natural continuous-time analog

$$M(t) \equiv \max\{K, \max\{S(s); 0 \leq s \leq t\}\} \quad (12)$$

and consequently the value converges from below (by the monotone convergence theorem) to its natural continuous-time version

$$E^* \left[\frac{S(T)}{B(T)} \left(1 - \frac{K}{M(T)} \right) + \int_0^T \frac{1}{B(t)} \left(1 - \frac{K}{M(t)} \right) dD(t) \right], \quad (13)$$

which is the same as Equation (8) except with the continuous process M substituted for M^n .

Consider instead the alternative formula of Equation (11). The sum in this expression can be interpreted as the approximating term in the definition of a Riemann–Stieltjes integral, and in the limit we have

$$E^* \left[\int_{0-}^T \frac{1}{B(t)} \frac{K}{M(t-)} dM(t) \right], \quad (14)$$

or, setting out separately the possible jump in M at $t = 0$, where $M(0) - M(0-) = (S(0) - K)^+$, we have the equivalent expression,

$$(S(0) - K)^+ + E^* \left[\int_0^T \frac{1}{B(t)} \frac{K}{M(t-)} dM(t) \right]. \quad (15)$$

At this point we add the assumption that any jumps in the process S are downward jumps, that is, $S(t) - S(t-) < 0$. This assumption implies that M is continuous: M can only jump up where S does and S cannot, while M is a cumulative maximum and therefore cannot jump down. It is nice that the assumption we need is also exactly the assumption that accommodates predictable dividend dates (which are times when the stock price can jump down), provided reinvesting dividends results in a continuous wealth process. This assumption rules out important discrete events (e.g., a merger announcement that causes the stock price to jump up 40%). From the continuity of M , $dM(t)/M(t-) = d \log(M)$, and defining $m(t) \equiv \log(M(t)/M(0))$, we have the alternative valuation expression

$$(S(0) - K)^+ + KE^* \left[\int_0^T \frac{1}{B(t)} dm(t) \right]. \quad (16)$$

Integration by parts and interchanging the order of integration gives

$$(S(0) - K)^+ + K \left(E^* \left[\frac{1}{B(T)} m(T) \right] - E^* \left[\int_0^T m(t) d \frac{1}{B(t)} \right] \right), \quad (17)$$

which is the formula that will allow us to derive a simple expression for the Black–Scholes case with dividends.

5. Black–Scholes Case With Dividends

In this section we consider the Black and Scholes (1973) case with possible continuous proportional dividends. We assume a constant positive interest rate r , so bond prices follow

$$B(t) = e^{rt}. \quad (18)$$

With the Black–Scholes assumption of a constant volatility per unit time and continuous proportional dividends, the stock price and cumulative dividend processes follow

$$S(t) = S(0) \exp\left(\int_0^t \left(\mu(u) - \frac{\sigma^2}{2} - \delta\right) du + \int_0^t \sigma dZ(u)\right) \quad (19)$$

and

$$D(t) = \int_0^t \delta S(u) du, \quad (20)$$

where $r, \sigma > 0$, and $\delta > 0$ are constants, the mean return process $\mu(t)$ is “arbitrary” (in quotes because it cannot be so wild that it generates arbitrage, e.g., by forcing the terminal stock price to a known value), and $Z(t)$ is a standard Wiener process. Under the risk-neutral probabilities P^* , the form of the process is the same but the mean return on the stock is r .

The following proposition gives formulas for the value and hedge ratio of the reload option. Given that there are very good uniform formulas (in terms of polynomials and exponentials) for the cumulative normal distribution function, the valuation and hedging formulas can be computed using two-dimensional numerical integration.

Proposition 1. *Suppose stock and bond returns are given by Equations (18)–(20) (the Black–Scholes case with dividends) and the current stock price is $S(0)$. Consider a reload option with current strike price K and remaining time to maturity τ . Its value is*

$$(S(0) - K)^+ + K \left(e^{-r\tau} E^*[m(\tau)] + r \int_0^\tau e^{-rt} E^*[m(t)] dt \right), \quad (21)$$

where the cumulative distribution function of $m(t)$ is given by $P^*\{m(t) \leq y\} = 0$ for $y < 0$ and by

$$P^*\{m(t) \leq y\} = \Phi\left(\frac{y - b - \alpha t}{\sigma\sqrt{t}}\right) - \exp\left(\frac{2\alpha(y - b)}{\sigma^2}\right) \Phi\left(\frac{-y + b - \alpha t}{\sigma\sqrt{t}}\right) \quad (22)$$

for $y \geq 0$, where $b \equiv -(\log(K/S(0)))^+$, $\alpha \equiv r - \delta - \frac{\sigma^2}{2}$, and $\Phi(\cdot)$ is the unit normal cumulative distribution function. The reload option’s replicating portfolio holds

$$\frac{K}{S(0)} \left(e^{-r\tau} P^*(m(\tau) > 0) + r \int_0^\tau e^{-rs} P^*(m(s) > 0) ds \right) \quad (23)$$

shares. Note that this hedge ratio and the valuation formula of Equation (21) are both per option currently held, and do not adjust for the decreasing number of options held when there is exercise. The hedge ratio does not include the initial grant of shares.

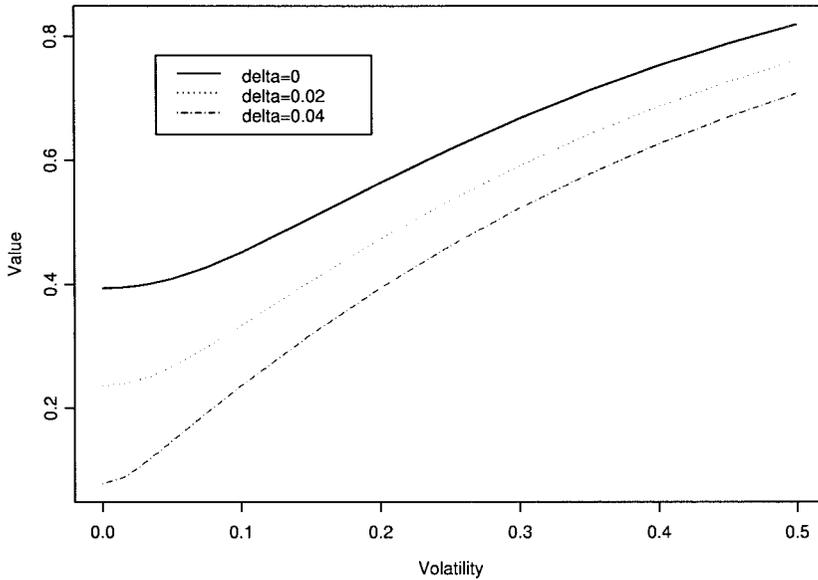


Figure 4
Reload option values for various volatilities and dividend rates

This shows the value of a par reload option with 10 years to maturity and a strike of \$1.00 as a function of the volatility (annual standard deviation) for three different annual dividend payout rates (0, 0.2, and 0.4), assuming an annual interest rate of 5%. As for an ordinary call option, the reload's value is increasing in volatility and decreasing in the dividend payout rate.

Before turning to the proof of Proposition 1, we direct the reader to Figures 4 and 5 which show values of reload options for various parameters, while Figures 6 and 7 compare the values of the reload option to those of a European call option. These figures confirm that the reload option value is increasing in σ and decreasing in δ . From Figure 6 we see that the reload option value for a non-dividend-paying stock is quite close to that of the European call option for low volatility, but as the volatility increases, there is a widening spread between the reload option value and the European call value. For volatilities much larger than are shown, the two must converge again, since both converge to the stock price as volatility increases. In Figure 7 we see that for a dividend-paying stock, the reload option value is uniformly higher than the European call option, as would be the value of an American call option.

The hedge ratio or, equivalently, the first derivative of the value with respect to the stock price, is always strictly positive and less than or equal to one. The hedge ratio is equal to one precisely when the option is at the money. Having a hedge ratio of ± 1 at the exercise boundary is familiar for American put and call options, and is an implication of the smooth-pasting conditions. The reason for the hedge ratio of one in this model is also due to smooth pasting, but is slightly more subtle to understand because both

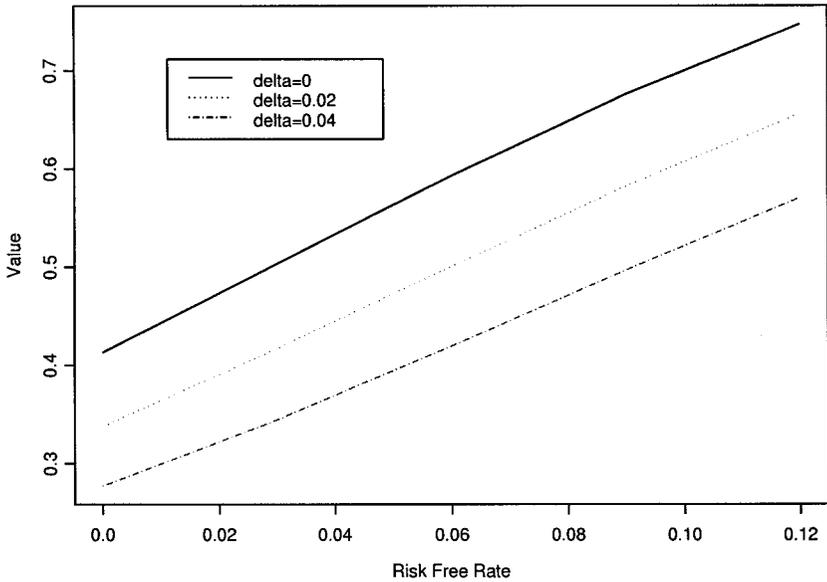


Figure 5
Reload option values for various interest and dividend rates

This shows the value of a par reload option with 10 years to maturity and a strike of \$1.00 as a function of the interest rate (annual number) for three different annual dividend payout rates (0, 0.2, and 0.4), assuming an annual standard deviation of 0.2. As for an ordinary call option, the reload's value is increasing in the interest rate and decreasing in the dividend payout rate.

the shares we get from exercise and the new reload options contribute to the hedge ratio. If we think of delaying exercise a short while, we will have the increase/decrease in the stock price on the net number of shares we get from exercising, and we will also have the same increase/decrease on the number of reload options (since the reload options are issued at-the-money with a hedge ratio of one). Since the number of new reload options plus the net number of new shares is equal to the number of old reload options, we can see that a hedge ratio of one is consistent with the usual smooth-pasting condition. Figure 2, which was introduced previously, shows the hedge ratio for the reload and European call options as a function of the ratio of stock price to strike price. In making this plot we included the initial grant of $(1 - \frac{K}{S(0)})$ shares for the in the money reload options. The hedge ratio is higher for a reload option than that for a European call option. As the reload option moves out of the money, however, the reload option hedge ratio looks closer to that of the European call option. Careful inspection of Figure 2 also suggests that the derivative of the hedge ratio for at the money reload options does not exist.

Figure 8 shows the gamma (the second derivative of the value with respect to stock price) for the reload and European call options as a function of S/K . Here we clearly see gamma is discontinuous at the money, just as an

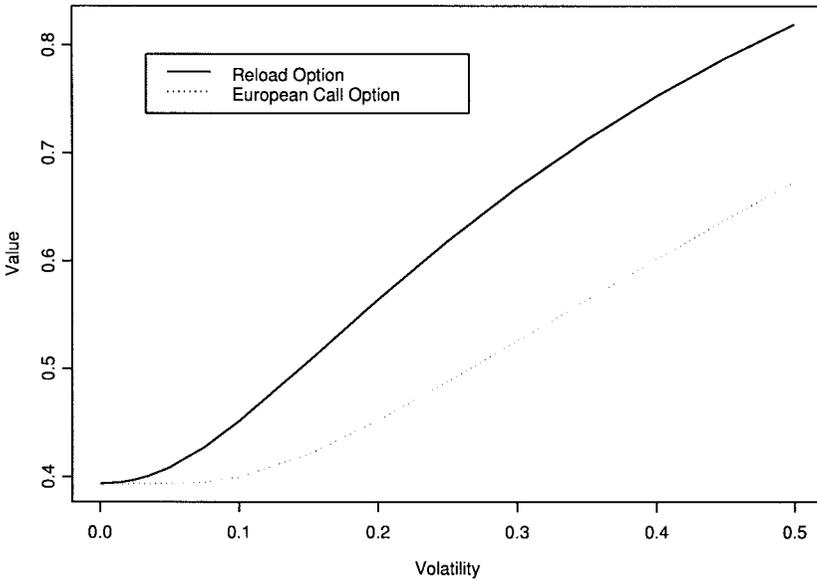


Figure 6
Comparison of reload option values with a Black–Scholes European call option: no dividends
 This shows the value of a par reload option (upper curve) and European call (lower curve) with 10 years to maturity and a strike of \$1.00 as a function of the volatility (annual standard deviation) when there are no dividends, assuming an annual interest rate of 5%. The two values move further apart as volatilities increase over the range shown, but both asymptote to \$1.00 (the stock price) asymptotically.

ordinary American option has a discontinuous gamma at the optimal exercise boundary. We also see that the gamma is somewhat lower for reload options which are closer to the money than that for a European call, and as the reload option goes out of the money, the gamma is somewhat higher but quite similar to that of the European call option.

In principle it is hard to assess the incentive effects of different compensation packages. However, as a first approximation we can look at the value, hedge ratio, and gamma of the reload option and compare those to those for a European call option. Figures 1, 2, 3, and 8 suggest that in terms of these measures, a reload option is quite similar to a European call. However, we should also consider the dynamic nature of the reload package. As higher stock prices are attained (i.e., as $M(t)$ increases), the reload option holdings decrease as more exercises occur. Assuming the shares received on exercise are held, the hedge ratio and gamma of the portfolio of $\frac{K}{M(t)}$ -remaining reload options plus $(1 - \frac{K}{M(t)})$ shares behave more like a share and less like an option. On the other hand, if not many exercises occur (i.e., $M(t)$ is close to K), and the reload option is out of the money, then the portfolio of shares plus options has a hedge ratio and gamma similar to a European call option.

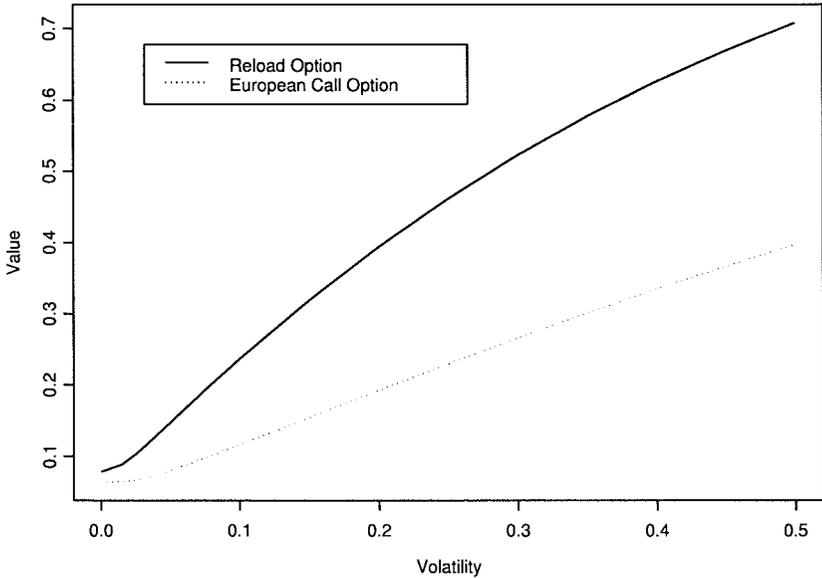


Figure 7

Comparison of reload option values with a Black-Scholes European call option: 4% dividends

This shows the value of a par reload option (upper curve) and European call (lower curve) with 10 years to maturity and a strike of \$1.00 as a function of the volatility (annual standard deviation) when there are 4% annual dividends, assuming an annual interest rate of 5%. The two values move further apart more quickly than without dividends as volatilities increase, and in fact the reload option asymptotes to a higher value. However, an American call would asymptote to the same value (the stock price).

In sum, the reload option seems similar to a hybrid of a share grant and a European call option.

Proof of Proposition 1. Assume without loss of generality that $\mu = r$, so that $P^* = P$ and no change of measure is needed. First, note that Equation (21) is obtained by substituting Equation (18) into Equation (17). [Recall that Equation (17) assumed $S(t)$ has no upward jumps, which is true here because $S(t)$ defined by Equation (19) is continuous.] Thus we see from Equations (21) and (19) that the value depends on the distribution of an expected maximum of a Wiener process with drift. Specifically, define $\alpha \equiv r - \delta - \frac{\sigma^2}{2}$ and

$$\begin{aligned} n(t) &\equiv \max_{0 \leq s \leq t} \log(S(s)/S(0)) \\ &= \max_{0 \leq s \leq t} \alpha t + \sigma Z(t). \end{aligned} \quad (24)$$

Then $m(t) = \max(n(t) + (\log(K/S(0)))^+, 0)$, and the distribution of $n(t)$ is well known: see, for example, Harrison (1985, Corollary 7 of Chapter 1,

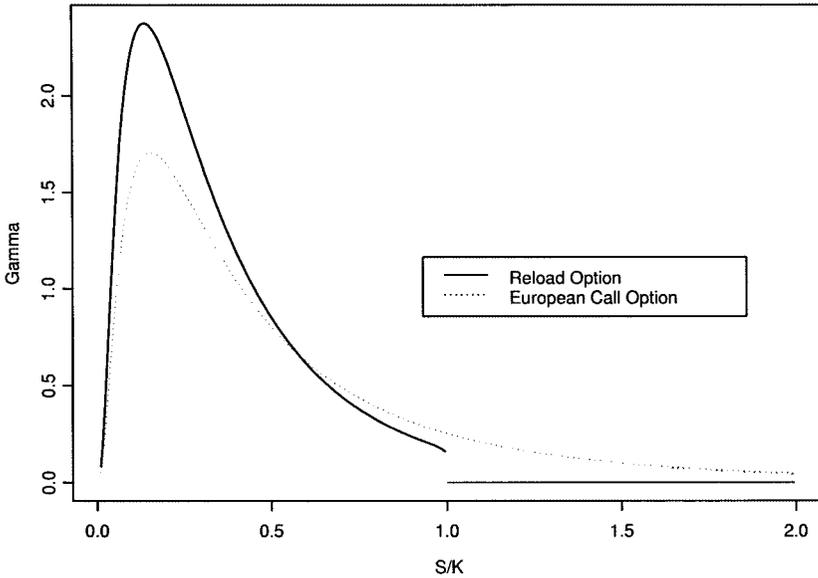


Figure 8
Comparison of reload option γ and European call option γ
 This shows the γ of a reload option (dark curve) and a European call option (dashed curve) with 10 years to maturity as a function of S/K when there are no dividends, the annual interest rate is 5%, the strike price is 1, and volatility is 30%.

Section 8). Specifically,

$$P\{n(t) \leq y\} = \begin{cases} 0 & \text{if } y < 0 \\ \Phi\left(\frac{y-at}{\sigma\sqrt{t}}\right) - \exp\left(\frac{2ay}{\sigma^2}\right)\Phi\left(\frac{-y-at}{\sigma\sqrt{t}}\right) & \text{if } y \geq 0, \end{cases}$$

where $\Phi(\cdot)$ is the unit normal distribution function. The claimed form of m 's distribution function [in Equation (22) and associated text] follows immediately.

It remains to derive the hedging formula of Equation (23). The hedge ratio (“delta”) of the reload option is the derivative of the value, exclusive of the first term in Equation (21) which is received up front and doesn’t need to be hedged, with respect to the stock price.⁵ From Equation (21) we can see that the hedge ratio will depend on the derivative of $E[m(t)]$ with respect to $S(0)$. It is convenient to compute $E[m(t)]$ using an integral over the density of $n(t)$, since the density of $m(t)$ depends on $S(0)$, while the density of $n(t)$

⁵ Some readers may be surprised to think of the hedge ratio as the simple derivative of the value with the stock price in the context of this complex seemingly path-dependent option. However, in between exercises, a reload option’s value is a function of the stock price and time, just like a call option or a European put option in the Black–Scholes world.

does not. Letting $\Psi(y)$ be the cumulative distribution function for $n(t)$, we have that

$$\begin{aligned} \frac{\partial}{\partial S(0)} E[m(t)] &= \frac{\partial}{\partial S(0)} \int_{n=(\log(K/S(0)))^+}^{\infty} (n - \log(K/S(0))) d\Psi(n) \\ &= \frac{1}{S(0)} \int_{n=(\log(K/S(0)))^+}^{\infty} d\Psi(n) \\ &= \frac{1}{S(0)} P(m(t) > 0). \end{aligned} \tag{25}$$

This expression is all we need to show that Equation (23) is the derivative of Equation (21) exclusive of the first term with respect to $S(0)$. ■

6. Time Vesting

In many cases the employee is prohibited from exercising the reload option until the end of an initial vesting period. Typically the reload options received after the initial exercise are also subject to the same vesting period. For example, reload options recently granted by Texaco have a vesting period of six months.⁶ In this section we can no longer rely on dominance arguments alone, so we assume that the employee's valuation is the same as the market's, that is, that the employee maximizes Equation (9) or equivalently Equation (7). Given the proximity of the solution of this problem to the solution in our base case, we expect this is a good approximation. We have two approaches to analyzing options with time vesting. The indirect approach approximates the option value using a lower bound based on restricting exercise to multiples of the vesting period from maturity. The direct approach uses a trinomial model with two state variables: the moneyness of the option and the amount of time the option has been vesting. The indirect approach is simpler and may be adequate for many purposes. The direct approach can be used to compute the option value and optimal exercise boundary to arbitrary precision, but only for specific stock price processes that can be approximated by a recombining trinomial.⁷

A useful upper bound is the value we have obtained for continuous exercise. A useful lower bound—and also a useful approximation to the value—is the value we have obtained for discrete exercise, provided that the time interval between adjacent dates t_{i-1} and t_i is (except perhaps the first interval) equal to the vesting period. For example, if the vesting period is 6 months

⁶ In some cases, the initial grant will vest differently than subsequent reload options. Furthermore, some firms have performance requirements to receive a reload option on exercise. We do not address these issues here, but they can be incorporated easily into the trinomial model discussed in this section.

⁷ While it is in principle possible to add state variables to approximate any process, our experience with this model suggests that will require computers with much more memory than current computers and/or some innovation in the computation (e.g., from truncating parts of the tree that do not contribute much to the value).

and we are 20 months from maturity, we assume we can exercise the option 2, 8, 14, and 20 months from now. This is a lower bound to the value given time vesting, since exercise at these dates is feasible given the vesting restriction, although it may be optimal to exercise on some other dates too. For this restrictive case, we know from Section 3 that it is optimal to follow the policy of exercising whenever the reload option is in the money. Thus we can bound the value of the reload option by the unrestricted case considered previously and the value computed in the overly restricted case. We think of this lower bound (which is the value with exercise at multiples of the vesting period from the end) as being a good approximation to the actual optimum, and general in the sense that we can use simulation to compute the option values for any stock price process. To evaluate our claim that this is a good approximation for most purposes, we turn to analysis that produces accurate valuation in a trinomial model.

To obtain a more accurate evaluation of a reload option with time vesting, we use a trinomial model which can be viewed as an approximation to the continuous time model. The trinomial model is similar to the binomial model of Cox, Ross, and Rubinstein (1979), but allows the stock to go up, down, or stay the same. This avoids some artificial even period/odd period irregularity of the exercise boundary in the binomial model.⁸ There are two state variables (not counting time) in our trinomial model: the ratio of the stock price to the reload's strike price, and the number of periods of vesting. The stock price itself is not needed as a state variable, since increasing the stock price and strike price in proportion also increases the reload option value in the same proportion. To keep the number of nodes manageable, we choose up and down movements for the stock that multiply to one.

To illustrate the trinomial model, here is a sketch of how it works in the simplest case without dividends. Over the time interval Δt , the stock price is increased by a factor $u = 1 + \sigma\sqrt{3\Delta t}/2$, stays the same, or is reduced by a factor $d = 1/u$. We take the risk-neutral probability of staying the same to be $\pi_m^* = 1/3$ and we take the risk-neutral probabilities π_u^* and π_d^* of going up or down to be what they need to be to make the mean stock return equal to the riskless rate. As Δt shrinks, the risk-neutral probabilities of the three states all converge to $1/3$, and the variance of the stock return is approximately $\sigma^2\Delta t$, as is desired.

Now, let $v_{i,k,m}$ be the option value per unit of strike when we are m time intervals (of length Δt each) from maturity, for the i th stock price node, and when we have been vested k periods (with the convention that vesting of the required K or more periods is labeled as K). At maturity ($m = 0$), the value is given by the option value. Letting $s_{i,m} = u^{i-1-m}$ be the stock price per unit of

⁸ There is a conceptual difference between the binomial and trinomial models, namely that the stock and bond span all the claims in the binomial model but not in the trinomial model. However, this does not matter to us because pricing in each model converges to pricing in the continuous model as we increase the number of periods per year.

strike at node (i, k, m) , then we have at maturity that $v_{i,k,0} = \max(s_{i,0} - 1, 0)$. Before maturity ($m > 0$), when we are not yet vested ($k < K$), the value of the option is given by

$$v_{i,k,m} = \frac{\pi_u^* v_{i+2,k+1,m-1} + \pi_m^* v_{i+1,k+1,m-1} + \pi_d^* v_{i,k+1,m-1}}{1 + r\Delta t}.$$

Before maturity ($m > 0$), if we are fully vested ($k = K$), then the value is the larger of the value from exercising now or not:

$$v_{i,K,m} = \max\left(s_{i,m} - 1 + v_{m+1,K,m}, \frac{\pi_u^* v_{i+2,K,m-1} + \pi_m^* v_{i+1,K,m-1} + \pi_d^* v_{i,K,m-1}}{1 + r\Delta t}\right). \tag{26}$$

In this expression, $s_{i,m} - 1$ is the value we get from exercising the option, and $v_{m+1,K,m}$ is the value of the new reload options issued at par. There is no adjustment for the number of options; the number issued ($1/s_{i,m}$ per option before reload) is just canceled by the increase in strike (by a factor $s_{i,m}$).

The optimal exercise boundary in the trinomial model is shown in Figure 9. In viewing the picture we should keep in mind that even with 1000 periods

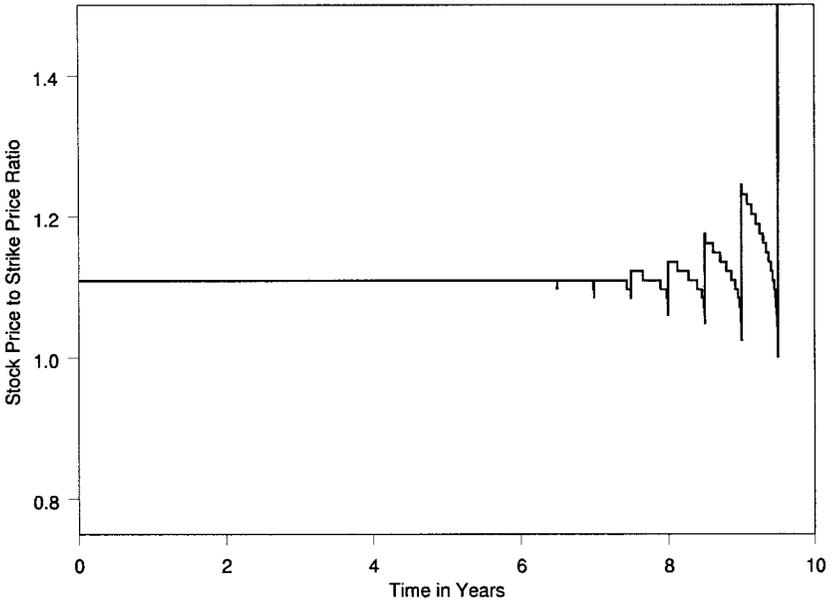


Figure 9
Optimal exercise boundary

This optimal exercise boundary for a vested reload option with 10 years to maturity and a six-month vesting period was computed using a trinomial model with 1000 periods per year, assuming an underlying non-dividend-paying stock with standard deviation of 30%/year and an interest rate of 5%/year. The vertical steps are due to the discrete set of possible stock prices. During the last half-year, the option is equivalent to a European call and is never exercised.

Table 1
Quality of the approximation: six-month time vesting

Volatility	Lower bound	Trinomial ($n = 200$)	Trinomial ($n = 400$)	Trinomial ($n = \infty$)	Upper bound
.3	.6285	.6292	.6302	.6326	.6676
.4	.7126	.7125	.7140	.7176	.7524

A lower bound, computable through simulation for general stock price processes, is a useful approximation to the more precise, but more limited trinomial model. For these numbers, options and their reloads vest in six months. The computations assume a maturity of 10 years, standard deviation of 40%/year, an annual interest rate of 5%, and no dividends. The parameter n is the number of periods per year, and $n = \infty$ is a reliable extrapolation to the limit as n increases without bound.

per year (as used in the computations for the figure), the discreteness in the price grid shows up as steps in the boundary. Qualitatively this is what is happening in the figure. The original option has 10 years to maturity and becomes vested in 6 months (as does each reloaded option, which can be reloaded again and again). During the last six months, the option (or any reload) is not exercised, since it is equivalent to an American call option (which is never optimally exercised before maturity when the stock pays no dividends). At later dates, exercise is most attractive when there is a multiple of the vesting period left in the option's life. Early in the option's life, the prospect of subsequent exercise on dates that are not a multiple of the vesting period from the end becomes more likely, and the optimal exercise boundary becomes flatter. (It is exactly flat on early dates in the figure, but this is only because of the discreteness of the prices on the vertical axis.) We expect that the exercise boundary will be similar, but smooth, between multiples of the vesting period from the end in the continuous model.

The two different approaches to valuation are compared in Tables 1 and 2. In both cases, using the lower bound (computed here by Monte Carlo simulation) provides an approximation to the value that may be acceptable for some purposes, especially given that the approximation error may well be smaller than the error due to estimation error in the volatility parameter input. In the table, n denotes the number of subperiods per year; the column $n = \infty$ comes from extrapolation and the very good theoretical approximation that the error for small Δt is proportional to $\sqrt{\Delta t}$. Although we know from above that always exercising when in the money at multiples of the vesting period is not optimal, the quality of the approximation shows that it is nearly optimal.

Table 2
Quality of the approximation: one-year time vesting

Volatility	Lower bound	Trinomial ($n = 200$)	Trinomial ($n = 400$)	Trinomial ($n = \infty$)	Upper bound
.3	.6144	.6140	.6151	.6178	.6676
.4	.6966	.6970	.6985	.7021	.7524

This table shows the quality of the lower bound as an approximation to the option value when the vesting period is one year. Other assumptions are as in Table 1.

7. Conclusion

Employee reload options are increasingly important. It is not possible to do a definitive analysis of the impact of reload options without including the context of the rest of the compensation package, including parts that are hard to quantify, such as the board's practice in granting new employee stock options in the absence of a right to reload. Nevertheless, we hope our theoretical and numerical results will help to rationalize the debate about the merits of reload options. We feel we have been successful in countering some of the wilder claims made by critics of reload options, and we have also given a scientific basis for valuation of the options for financial reporting.

References

- Arnason, S., and R. Jagannathan, 1994, "Evaluating Executive Stock Options Using the Binomial Option Pricing Model," working paper, University of Minnesota.
- Beaglehole, D., P. H. Dybvig, and G. Zhou, 1997, "Going to Extremes: Correcting Simulation Bias in Exotic Option Valuation," *Financial Analysts Journal*, January–February, 62–68.
- Black, F., and M. Scholes, 1973, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81, 637–654.
- Cox, J., S. Ross, and M. Rubinstein, 1979, "Option Pricing: A Simplified Approach," *Journal of Financial Economics*, 7, 229–263.
- Financial Accounting Standards Board, 1995, "Accounting for Stock-Based Compensation," Statement of Financial Accounting Standards no. 123, Financial Accounting Standards Board.
- Frederic W. Cook and Company, 1998, "Reload Stock Options—The First Ten Years," working paper, Frederic W. Cook and Company.
- Gay, C., 1999, "Hard to Lose," *Wall Street Journal*, April 8, R6.
- Harrison, J. M., 1985, *Brownian Motion and Stochastic Flow Systems*, Wiley, New York.
- Hemmer, T., S. Matsunaga, and T. Shevlin, 1998, "Optimal Exercise and the Cost of Granting Employee Stock Options With a Reload Provision," *Journal of Accounting Research*, 36, 231–255.
- Jagannathan, R., 1984, "Call Options and the Risk of Underlying Securities," *Journal of Financial Economics*, 13, 425–434. See also Jagannathan, R., 1985, "Errata [Call Options and the Risk of Underlying Securities]," *Journal of Financial Economics*, 14, 323.
- Karatzas, I., and Shreve, S. E., 1991, *Brownian Motion and Stochastic Calculus*, 2nd ed., Springer-Verlag, New York.
- Reingold, J., and L. N. Spiro, 1998, "Nice Option if You Can Get It," *Business Week*, May 4, 111–112.
- Saly, P. J., R. Jagannathan, and S. J. Huddart, 1999, "Valuing the Reload Features of Executive Stock Options," *Accounting Horizons*, 13, 219–240.