Increases in Risk Aversion and Portfolio Choice in a Complete Market

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Abstract

We examine the effect of changes in risk aversion on optimal portfolio choice in a complete market. We show that an agent who is less risk averse than another chooses a portfolio whose payoff is distributed as the other’s payoff plus a nonnegative random variable plus conditional-mean-zero noise. Conversely, an agent who chooses a portfolio whose payoff in all complete markets, is distributed as another’s payoff plus a nonnegative random variable plus conditional-mean-zero noise is less risk averse than the other. In this sense, the two portfolio choices represent a trade-off between risk and return, and not only in a mean-variance world. In addition, if either agent has non-increasing absolute risk aversion, then the non-negative random variable can be chosen to be a constant.

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I. Introduction

The trade-off between risk and return arises in many portfolio problems in finance. This trade-off is more-or-less assumed in mean-variance optimization,¹ and is also present in the comparative statics for two-asset portfolio problems explored by Arrow (1965) and Pratt (1964) (for a model with a riskless asset) and Kihlström, Romer, and Williams (1981) and Ross (1981) (for a model without a riskless asset). However, the trade-off is less clear in portfolio problems with many risky assets, as pointed out by Hart (1975). Assuming a complete market with many states (and therefore many assets), we show that a less risk-averse (in the sense of Arrow and Pratt) agent’s portfolio payoff is distributed as the payoff for the more risk-averse agent, plus a non-negative random variable (extra return), plus conditional-mean-zero noise (risk). Therefore, the general complete-markets portfolio problem, which may not be a mean-variance problem, still trades off risk and return.

We also provide several related results. One is a converse theorem. Suppose there are two agents, such that in all complete markets, the first agent chooses a payoff that is distributed as the second’s payoff, plus a non-negative random variable, plus conditional-mean-zero noise. Then the first agent is less risk averse than the other agent. We also prove a multiple-period result. Consumption at each date may not be ordered when risk aversion changes, due to shifts in the timing of consumption. However, for agents with the same pure rate of time preference, there is a weighting of probabilities across periods that preserves the single-period result. We also show that, if either agent has non-increasing absolute risk aversion, then the non-negative random variable can be chosen to be a constant. We also give a counter-example that shows that in general, the non-negative random variable cannot be chosen to be a constant. In this case, the less risk averse agent’s payoff can have a higher mean and a lower variance than the more risk averse agent’s payoff.² This can happen because

¹In a mean-variance world with multivariate normality asset payoffs and increasing concave von Neumann-Morgenstern utility functions, a less risk averse agent will choose a higher mean and higher variance. Given joint normality, this implies that the less risk averse agent’s payoff is distributed as the other payoff, plus a non-negative constant plus conditional-mean zero noise.

²As it is already known in the literature, variance is not a good risk measure for von Neumann-Morgenstern utility functions.
although adding condition-mean-zero noise always increases variance, adding the non-negative random variable decreases variance if it is sufficiently negatively correlated with rest.

The proofs in the paper use results from stochastic dominance, complete-markets portfolio choice, and Arrow-Pratt risk aversion. One contribution of the paper is to show how these concepts relate to each other. We use general versions of the stochastic dominance results for $L^1$ random variables and monotone concave preferences, following Strassen (1965) and Ross (1971). To see why our results are related to stochastic dominance, note that if the first agent’s payoff equals the second agent’s payoff plus a non-negative random variable plus conditional-mean-zero noise, this is equivalent to saying negative the first agent’s payoff is monotone-concave dominated by negative the second agent’s payoff.

Section II introduces the model setup and gives examples, Section III develops the basic framework and derives the main results. Section IV extends the main results in a multiple-period model. Section V illustrates the main results using some examples and Section VI concludes.

II. Model Setup and Examples

We want to work in a fairly general setting with complete markets and strictly concave increasing von Neumann-Morgenstern preferences. There are two agents $A$ and $B$ with von Neumann-Morgenstern utility functions $U_A(c)$ and $U_B(c)$, respectively. We assume that $U_A(c)$ and $U_B(c)$ are of class $C^2$, $U'_A(c) > 0$, $U'_B(c) > 0$, $U''_A(c) < 0$ and $U''_B(c) < 0$. Each agent’s problem has the form:

Problem 1 Choose random consumption $\tilde{c}$ to

$$\max E[U_i(\tilde{c})],$$

$$s.t. \ E[\tilde{\rho}\tilde{c}] = w_0. \quad (1)$$

In Problem 1, $i = A$ or $B$ indexes the agent, $w_0$ is initial wealth (which is the same for both agents), and $\tilde{\rho} > 0$ is the state price density. We will assume that agents
have optimal random consumptions, called \( \tilde{c}_A \) and \( \tilde{c}_B \).

The first order condition is \( U'(\tilde{c}) = \lambda \tilde{p} \), i.e., the marginal utility is proportional to the state price density \( \tilde{p} \). Therefore, the first order conditions for the two agents’ optima are given by

\[
U'_A(\tilde{c}_A) = \lambda_A \tilde{p}, \quad U'_B(\tilde{c}_B) = \lambda_B \tilde{p},
\]

i.e.,

\[
\tilde{c}_A = I_A(\lambda_A \tilde{p}), \quad \tilde{c}_B = I_B(\lambda_B \tilde{p}),
\]

where \( I_A \) and \( I_B \) are the inverse function of \( U'_A(\cdot) \) and \( U'_B(\cdot) \) respectively, by negativity of the second derivatives, \( \tilde{c}_A \) and \( \tilde{c}_B \) are decreasing functions of \( \tilde{p} \).

Our main result will be that \( \tilde{c}_A \sim_d \tilde{c}_B + \tilde{z} + \tilde{\epsilon} \), where \( \sim_d \) denotes “is distributed as”, \( \text{Prob}(\tilde{z} \geq 0) = 1 \), and \( E[\tilde{\epsilon}|c_B + \tilde{z}] = 0 \). Before presenting our main results, we consider two examples to illustrate this result.

**Example II.1** \( B \) is more risk averse than \( A \), \( A \) and \( B \) have the same initial wealth \( w_0 \) and the utility functions are as follows

\[
U_A(c) = -\frac{1}{2} \left( \frac{1}{a_1} - c \right)^2, \quad U_B(c) = -\frac{1}{2} \left( \frac{1}{a_2} - c \right)^2, \quad \frac{1}{a_i} - c > 0, \quad i = 1, 2. \quad a_1 < a_2.
\]

We assume that the state price density \( \tilde{p} \) is uniformly distributed in \([0, 1]\). The first order conditions give us

\[
\tilde{c}_A = \frac{1}{a_1} - \lambda_A \tilde{p}, \quad \tilde{c}_B = \frac{1}{a_2} - \lambda_B \tilde{p}.
\]

From the budget constraint \( E[\tilde{p}c] = w_0 \), (assuming \( a_1 < \frac{1}{2w_0} \) and \( a_2 < \frac{1}{2w_0} \)), we get that \( \lambda_A = 3 \left( \frac{1}{2a_1} - w_0 \right) \) and \( \lambda_B = 3 \left( \frac{1}{2a_2} - w_0 \right) \). Therefore, \( \tilde{c}_A \) is uniformly distributed in \( \left[ -\frac{1}{2a_1} + 3w_0, \frac{1}{a_1} \right] \) and \( \tilde{c}_B \) is uniformly distributed in \( \left[ -\frac{1}{2a_2} + 3w_0, \frac{1}{a_2} \right] \).

\[
E[\tilde{c}_A] - E[\tilde{c}_B] = \frac{1}{4} \left( \frac{1}{a_1} - \frac{1}{a_2} \right) > 0, \quad \text{and the support of } \tilde{c}_A \text{ is wider than that of } \tilde{c}_B, \text{ let } a_2 = \frac{6a_1}{3 + 6a_1 w_0}, \text{ i.e., the range of } \tilde{c}_A \text{ is twice of that of } \tilde{c}_B, \text{ and } \tilde{\epsilon} \text{ has a Bernoulli distribution drawn independently of } \tilde{c}_B \text{ with two equally possible outcomes } \frac{3}{2} \left( \frac{1}{2a_2} - w_0 \right) \text{ and } -\frac{3}{2} \left( \frac{1}{2a_2} - w_0 \right). \text{ It is not difficult to see that } \tilde{c}_A \text{ is distributed as } \tilde{c}_B + \tilde{z} + \tilde{\epsilon}, \text{ where } \tilde{z} = E[\tilde{c}_A] - E[\tilde{c}_B] \text{ and } E[\tilde{\epsilon}|c_B] = 0.\]
From this example, we can see that an agent who is less risk averse than another chooses a portfolio whose payoff is distributed as the other’s payoff plus a nonnegative random variable plus conditional-mean-zero noise. In this example, this nonnegative random variable $\tilde{z}$ can be chosen to be a constant, and therefore the variance of the less risk averse agent’s payoff is higher. In addition, in this example, the random noise $\tilde{\varepsilon}$ can be chosen to be independent of $\tilde{c}_B + \tilde{z}$. It is interesting to see whether the results hold for more general utility functions. Can the nonnegative random variable always be chosen to be a constant? Is the variance of the less risk averse agent’s payoff always higher? Can the random noise always be chosen to be independent of $\tilde{c}_B + \tilde{z}$? As we can see from the following example, the answers to these questions are negative.

**Example II.2** $B$ is more risk averse than $A$, $A$ and $B$ have the same initial wealth $w_0$ and the utility functions are as follows

$$U_A(c) = -\left(\frac{1}{a} - c\right)^{n_A}, U_B(c) = -\left(\frac{1}{a} - c\right)^{n_B}, \text{ where } \frac{1}{a} - c > 0, \ n_A < n_B.$$  

The first order conditions give us

$$U_A'(\tilde{c}_A) = n_A \left(\frac{1}{a} - \tilde{c}_A\right)^{n_A-1} = \lambda_A \tilde{\rho}, \ U_B'(\tilde{c}_B) = n_B \left(\frac{1}{a} - \tilde{c}_B\right)^{n_B-1} = \lambda_B \tilde{\rho}. \quad (5)$$

We will prove later in the main results and in Section V that $\tilde{c}_A \sim \tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$ and $E[\tilde{\varepsilon}|c_B + z] = 0$. However, we can see that $\tilde{z}$ may not be chosen to be a constant $E[\tilde{c}_A] - E[\tilde{c}_B] > 0$ (pick $\tilde{\rho}$, such that $E[\tilde{c}_A] \neq E[\tilde{c}_B]$). From previous discussion, we know that $\tilde{c}_A$ and $\tilde{c}_B$ are decreasing functions of $\tilde{\rho}$, from (5), it is not difficult to see that $\tilde{c}_A \to \frac{1}{a}$ and $\tilde{c}_B \to \frac{1}{a}$ when $\tilde{\rho} \to 0$. Letting $\tilde{c}_{A'} = \tilde{c}_A - (E[\tilde{c}_A] - E[\tilde{c}_B])$, then it is easy to see that there exists $\bar{c}$, such that for $c > \bar{c}$, we have $\tilde{c}_{A'} < \tilde{c}_B$. If $\tilde{c}_{A'} \sim \tilde{c}_B + \varepsilon$ and $E[\varepsilon|c_B] = 0$, then Prob($\varepsilon > 0$) > 0 for any given $c_B$ (we will see in Section 3 that $\varepsilon \neq 0$), therefore, esssup $\tilde{c}_{A'} = \inf\{a \in R : \mu(\{\tilde{c}_B + \varepsilon > a\}) = 0\} > \inf\{a \in R : \mu(\{\tilde{c}_B > a\}) = 0\} = \esssup \tilde{c}_B$. Contradiction! We conclude that we cannot write $\tilde{c}_{A'} \sim \tilde{c}_B + \varepsilon$, where $E[\varepsilon|c_B] = 0$, this implies that in this example, $\tilde{z}$ cannot be chosen to be a constant.
We now show that $\tilde{\varepsilon}$ cannot be chosen to be independent of $\tilde{c}_B + \tilde{z}$. If $\tilde{\varepsilon}$ is independent with $\tilde{c}_B + \tilde{z}$, then we have $\text{esssup} \tilde{c}_A \geq \text{esssup} \{\tilde{c}_B + \tilde{z}\} > \text{esssup} \{\tilde{c}_B + \tilde{z} + \tilde{\varepsilon}\}$. The left inequality is because $\tilde{z}$ is a nonnegative random variable and the right inequality is because $\text{Prob}(\tilde{\varepsilon} > 0) > 0$ for any given $\tilde{c}_B + \tilde{z}$ and $\tilde{\varepsilon} \neq 0$. Contradiction!

To find an example that the variance of the less risk averse agent’s payoff can be smaller, we assume that $\tilde{\rho}$ has a discrete distribution, i.e., $\tilde{\rho} = 0$ with probability $\frac{1}{2}$, $\tilde{\rho} = \frac{1}{4}$ with probability $\frac{1}{4}$ and $\tilde{\rho} = \frac{1}{2}$ with probability $\frac{1}{4}$. From (5), we have $\tilde{c}_A = \frac{1}{a} - \left(\frac{\lambda_A}{n_A} \tilde{\rho}\right)^{\frac{1}{n_A-1}}$. From the budget constraint, $E[\tilde{\rho} \tilde{c}_A] = w_0$, (assuming $a < \frac{3}{16w_0}$). It is not hard to get that $\text{Var}(\tilde{c}_A) = 0.421 \left(\frac{3}{16a} - 4w_0\right)^2 < \text{Var}(\tilde{c}_B) = 0.426 \left(\frac{3}{16a} - 4w_0\right)^2$ for $n_A = 3$ and $n_B = 5$, i.e., the variance of the more risk averse agent’s payoff is higher.

From these examples, we know that the less risk averse agent $A$’s payoff is distributed as the more risk averse agent $B$’s payoff plus a non-negative random variable plus conditional-mean-zero noise. In general, this non-negative random variable $\tilde{z}$ may not be chosen to be a constant and the random noise $\varepsilon$ may not be chosen to be independent of $\tilde{c}_B + \tilde{z}$. Interestingly, the variance of the more risk averse agent’s payoff can be higher though the mean of the payoff is always lower.

Next example shows that if the utility functions are not strictly concave, then our main results do not hold.

**Example II.3** $B$ is more risk averse than $A$, $A$ and $B$ have the same initial wealth $w_0 = 2$ and the utility functions are as follows

$$U_A(c) = \begin{cases} 3 - e^{2-c} & c < 2 \\ c & 2 \leq c \leq 4 \\ -\frac{1}{16}(c - 12)^2 + 8 & 4 < c < 8 \\ \frac{1}{2}c + 3 & 8 \leq c \leq 12 \\ 10 - e^{-(c-12)/2} & c > 12, \end{cases}$$

\(^3\)Having $\tilde{\rho} = 0$ with positive probability implies there is arbitrage, which is okay because the preferences are satiated, this is not an essential feature of the example. In Corollary 1, we provide a general condition under which a less risk averse agent’s payoff has a higher expected return and a smaller variance.
We assume that $\tilde{\rho} = \frac{1}{2}$ with probability $\frac{1}{2}$ and $\tilde{\rho} = \frac{1}{4}$ with probability $\frac{1}{2}$. Then, it is not difficult to see that $\tilde{c}_A = (4, 8)$ and $\tilde{c}_B = (2, 12)$ is the optimal consumption for agent $A$ and $B$ for $\lambda_A = \lambda_B = 2$. However, we cannot order $\tilde{c}_A$ and $\tilde{c}_B$.

We next present and prove our main results.

### III. Main Results

We now prove our main results. We compare agents’ risk aversion in the sense of Arrow and Pratt. Recall that we assume that agent $B$ is more risk averse than agent $A$, and that both utility functions are strictly increasing and strictly concave $C^2$ functions. By Pratt (1964), we have the concave transform characterization

$$U_B(c) = \begin{cases} \frac{5}{2} - \frac{1}{2}e^{4-2c} & c < 2 \\ c & 2 \leq c \leq 4 \\ -\frac{1}{16}(c-12)^2 + 8 & 4 < c < 8 \\ \frac{1}{2}c + 3 & 8 \leq c \leq 12 \\ 9\frac{1}{2} - \frac{1}{2}e^{-(c-12)} & c > 12. \end{cases}$$

We assume that $\rho = \frac{1}{2}$ with probability $\frac{1}{2}$ and $\rho = \frac{1}{4}$ with probability $\frac{1}{2}$. Then, it is not difficult to see that $\tilde{c}_A = (4, 8)$ and $\tilde{c}_B = (2, 12)$ is the optimal consumption for agent $A$ and $B$ for $\lambda_A = \lambda_B = 2$. However, we cannot order $\tilde{c}_A$ and $\tilde{c}_B$.

We next present and prove our main results.

**Lemma 1** If $B$ is more risk averse than $A$, $\left( \forall c, -\frac{U_B''(c)}{U_B'(c)} \geq -\frac{U_A''(c)}{U_A'(c)} \right)$, then

1. for any solution to (2) (which may not satisfy the budget constraint (1)), there exists some critical consumption level $c^*$ (can be $\pm \infty$) such that $\tilde{c}_A \geq \tilde{c}_B$ when $\tilde{c}_B \geq c^*$, and such that $\tilde{c}_A \leq \tilde{c}_B$ when $\tilde{c}_B \leq c^*$;

2. assuming $\tilde{c}_A$ and $\tilde{c}_B$ have finite means, and $A$ and $B$ have equal initial wealths, then $E[\tilde{c}_A] \geq E[\tilde{c}_B]$.

Proof: Using the concave transform characterization of more risk averse in (6), the first order condition (2) becomes

$$U_A'(\tilde{c}_A) = \lambda_A \tilde{\rho} = \frac{\lambda_A}{\lambda_B} \lambda_B \tilde{\rho} = \frac{\lambda_A}{\lambda_B} G''(U_A(\tilde{c}_B)) U_A'(\tilde{c}_B).$$

(7)
(7) can be written as

$$\tilde{c}_A = I_A \left[ \frac{\lambda_A}{\lambda_B} G'(U_A(\tilde{c}_B)) \cdot U'_A(\tilde{c}_B) \right].$$

(8)

Because marginal utility is strictly decreasing, we have

$$
\begin{align*}
&\begin{cases}
\tilde{c}_A > \bar{c}_B, & \text{if } G' < \frac{\lambda_B}{\lambda_A} \\
\tilde{c}_A = \bar{c}_B, & \text{if } G' = \frac{\lambda_B}{\lambda_A} \\
\tilde{c}_A < \bar{c}_B, & \text{if } G' > \frac{\lambda_B}{\lambda_A}.
\end{cases}
\end{align*}
$$

Choose $c^*$ so that $G'(U_A(c^*)) = \frac{\lambda_B}{\lambda_A}$ if possible, or pick $c^* = -\infty$ if $G' < \frac{\lambda_B}{\lambda_A}$ everywhere or $c^* = +\infty$ if $G' > \frac{\lambda_B}{\lambda_A}$ everywhere. If $\bar{c}_B \geq c^*$, then $G'(U_A(\bar{c}_B)) \leq G'(U_A(c^*)) = \frac{\lambda_B}{\lambda_A}$, i.e., $G' \leq \frac{\lambda_B}{\lambda_A}$, therefore, $\tilde{c}_A \geq \bar{c}_B$. If $\bar{c}_B \leq c^*$, then $G'(U_A(\bar{c}_B)) \geq G'(U_A(c^*)) = \frac{\lambda_B}{\lambda_A}$, i.e., $G' \geq \frac{\lambda_B}{\lambda_A}$, therefore, $\tilde{c}_A \leq \bar{c}_B$. This proves statement 1.

Now suppose that $A$ and $B$ have equal initial wealths, then the budget constraints for the agents are that

$$E[\rho \tilde{c}_A] = E[\rho \tilde{c}_B] = w_0,$$

(9) therefore, we have $E[\rho(\tilde{c}_A - \tilde{c}_B)] = 0$, because $\rho$ is inversely related to both $\tilde{c}_A$ and $\tilde{c}_B$, $\rho$ is smaller in the states with $\tilde{c}_B \geq c^*$, in which $\tilde{c}_A \geq \tilde{c}_B$ and $\rho$ is bigger in the states with $\tilde{c}_B \leq c^*$, in which $\tilde{c}_A \leq \tilde{c}_B$. Let $\rho^*$ be the critical value corresponding to $c^*$, i.e., $E[\rho^* c^*] = w_0$, then,

$$0 = E[\rho(\tilde{c}_A - \tilde{c}_B)] = E[\rho^*(\tilde{c}_A - \tilde{c}_B)] + E[(\rho - \rho^*)(\tilde{c}_A - \tilde{c}_B)] \leq \rho^* E[\tilde{c}_A - \tilde{c}_B].$$

Therefore, $E[\tilde{c}_A] \geq E[\tilde{c}_B]$. Q.E.D.

Lemma 1 gives us a sense in which decreasing the agent’s risk aversion takes us further from the riskless asset. In fact, we can obtain a more explicit description of how decreasing the agent’s risk aversion changes the optimal portfolio choice. The description and proof are both related to monotone concave stochastic dominance.\footnote{We avoid using the concept of second order stochastic dominance in this paper because there are two different definitions in the literature. In this paper, we follow Ross (1971): (1) if $E[V(X)] \geq E[V(Y)]$ for all nondecreasing utility functions, then $X$ monotonically stochastically dominates $Y$; (2) if $E[V(X)] \geq E[V(Y)]$ for all concave utility functions, then $X$ concavely stochastically dominates $Y$.}
The following theorem gives a distributional characterization of stochastic dominance for all monotone and concave utility functions of one random variable over another. The form of this result is from Ross (1971) and is a special case of a result of Strassen (1965) which generalizes a traditional result for bounded random variables to possibly unbounded random variables with finite means. Rothschild and Stiglitz (1972) popularized a similar characterization of stochastic dominance for all concave utility functions (which implies equal means) that is a special case of another result of Strassen’s.

**Theorem 1** Let $X$ and $Y$ be two random variables defined in $\mathbb{R}^1$ with finite mean; then

$$E[V(X)] \geq E[V(Y)],$$

for all concave nondecreasing utility function, $V(\cdot)$, iff

$$Y \sim X - Z + \varepsilon,$$

where $Z \geq 0$, and $E[\varepsilon|X - Z] = 0$.

Proof: *(Sufficiency)* The monotonicity of the utility function and Jensen’s inequality yield

$$E[V(Y)] = E[V(X - Z + \varepsilon)] = E[E[V(X - Z + \varepsilon)|X, Z]] \leq E[V(X - Z)]$$

$$= E[E[V(X - Z)|X]] \leq E[V(X)].$$

*(Necessity)* Let $\mu_1$ be the distribution of $-X$, and let $\mu_2$ be the distribution of $-Y$. From Theorem 9 in Strassen (1965), the following two statements are equivalent.

(i) For any concave nondecreasing function $V(s)$, $\int V(-s)d\mu_1(s) \geq \int V(-s)d\mu_2(s)$,

(ii) There exists a submartingale $\xi_n$, $n = 1, 2$, such that the distribution of $\xi_n$ is $\mu_n$, i.e., $E[\xi_2|\xi_1] \geq \xi_1$.

From (ii), we have (iii) $\xi_2 = \xi_1 + (E[\xi_2|\xi_1] - \xi_1) + (\xi_2 - E[\xi_2|\xi_1])$, letting $Z = \int \xi_2 - \xi_1$, and

(3) if $E[V(X)] \geq E[V(Y)]$ for all concave nondecreasing utility functions, then $X$ monotone concave stochastically dominates $Y$.  

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\[ E[\xi_2|\xi_1] - \xi_1 \geq 0, \text{ and } \varepsilon = -\xi_2 + E[\xi_2|\xi_1], \text{ we have} \]

\[ E[\varepsilon|\xi_1 + Z] = E[(-\xi_2 + E[\xi_2|\xi_1])|E[\xi_2|\xi_1]] = 0. \]

Then, from (i), we have \( E[V(X)] \geq E[V(Y)] \), from (iii), we have \( -Y \sim -X + Z - \varepsilon \), where \( Z \sim E[-Y|X] + X \) and \( \varepsilon \sim Y + E[-Y|X] \), we get \( Y \sim X - Z + \varepsilon \), where \( Z \geq 0 \) and \( E[\varepsilon|X - Z] = 0 \). \( Q.E.D. \)

Now we prove our main results.

**Theorem 2**

1. If \( B \) is weakly more risk averse than \( A \) in the sense of Arrow and Pratt, then \( c_A \) is distributed as \( \tilde{c}_B + \tilde{z} + \tilde{\varepsilon} \), where \( \tilde{z} \geq 0 \) and \( E[\tilde{\varepsilon}|c_B + \tilde{z}] = 0 \).

   Furthermore, whenever \( \tilde{c}_A \neq \tilde{c}_B \), neither \( \tilde{z} \) nor \( \tilde{\varepsilon} \) is identically zero.

2. Conversely, if for all distributions of \( \tilde{\rho} \), \( \tilde{c}_A \) is distributed as \( \tilde{c}_B + \tilde{z} + \tilde{\varepsilon} \), where \( \tilde{z} \geq 0 \) and \( E[\tilde{\varepsilon}|c_B + \tilde{z}] = 0 \), then \( B \) is weakly more risk averse than \( A \).

**Proof:** Firstly, we prove result 1. The first step of the proof is to show that \(-\tilde{c}_B\) monotone concave stochastically dominates \(-\tilde{c}_A\). By Lemma 1, \( \tilde{c}_A \) and \( \tilde{c}_B \) are monotonically related and there is a critical value \( c^* \) above which \( \tilde{c}_A \) is weakly larger and below which \( \tilde{c}_B \) is weakly larger. We have \( \{-\tilde{c}_A \geq q\} \supseteq \{-\tilde{c}_B \geq q\} \) when \( q \geq -c^* \). This is because if \( -\tilde{c}_B \geq q \) and \( q \geq -c^* \), then \( \tilde{c}_B \leq -q \leq c^* \), from Lemma 1, we have \( \tilde{c}_A \leq \tilde{c}_B \leq -q \), i.e., \( -\tilde{c}_A \geq q \). Similarly, we have \( \{-\tilde{c}_A \leq q\} \supseteq \{-\tilde{c}_B \leq q\} \) when \( q \leq -c^* \). By the definition of the distribution functions for \( -\tilde{c}_A \) and \( -\tilde{c}_B \), this means precisely that

\[ F_{-\tilde{c}_A}(q) \geq F_{-\tilde{c}_B}(q) \text{ for } q \leq -c^*, \text{ and } F_{-\tilde{c}_A}(q) \leq F_{-\tilde{c}_B}(q) \text{ for } q \geq -c^*. \]  \( (10) \)

Define

\[ I(c) = \int_{q=-\infty}^{c} [F_{-\tilde{c}_A}(q) - F_{-\tilde{c}_B}(q)] dq. \]  \( (11) \)

\( I(+\infty) = E[\tilde{c}_A - \tilde{c}_B] \geq 0 \) by integration by parts and Lemma 1. It is not hard to see

\[ E[V(-\tilde{c}_B) - V(-\tilde{c}_A)] = \int_{-\infty}^{+\infty} V(q)[dF_{-\tilde{c}_B}(q) - dF_{-\tilde{c}_A}(q)] \]
\[
\begin{align*}
&= \lim_{q \to +\infty} V(q)[F_{\tilde{c}_B}(q) - F_{\tilde{c}_A}(q)] - \lim_{q \to -\infty} V(q)[F_{\tilde{c}_B}(q) - F_{\tilde{c}_A}(q)] \\
&\quad - \int_{-\infty}^{+\infty} V'(q)[F_{\tilde{c}_B}(q) - F_{\tilde{c}_A}(q)]dq = - \int_{-\infty}^{+\infty} V'(q)[F_{\tilde{c}_B}(q) - F_{\tilde{c}_A}(q)]dq, \quad (12)
\end{align*}
\]

where the limits are zero because \(c_A\) and \(c_B\) have finite mean and \(V(\cdot)\) is concave nondecreasing. Now, (12) is equal to

\[
\begin{align*}
&\quad - \int_{-\infty}^{+\infty} V'(-c^*)[F_{\tilde{c}_B}(q) - F_{\tilde{c}_A}(q)]dq - \int_{-c^*}^{-\infty} (V'(q) - V'(-c^*))[F_{\tilde{c}_B}(q) - F_{\tilde{c}_A}(q)]dq \\
&\quad - \int_{-c^*}^{+\infty} (V'(q) - V'(-c^*))[F_{\tilde{c}_B}(q) - F_{\tilde{c}_A}(q)]dq. \quad (13)
\end{align*}
\]

From the concavity of \(V(\cdot)\) and (10), it is easy to see that the second and third terms in (13) are nonnegative. From monotonicity of \(V(\cdot)\) and \(I(+\infty) \geq 0\), we also get that the first term of (13) is nonnegative. Therefore, we get

\[
E[V(-\tilde{c}_B)] \geq E[V(-\tilde{c}_A)]. \quad (14)
\]

By Theorem 1, this says that \(-\tilde{c}_A\) is distributed as \(-\tilde{c}_B - \tilde{\varepsilon} + \tilde{\varepsilon}\), where \(\tilde{\varepsilon} \geq 0\) and \(E[\tilde{\varepsilon}] - c_B - \tilde{z} = 0\). This is exactly the same as saying that \(\tilde{c}_A\) is distributed as \(\tilde{c}_B + \tilde{z} + (-\tilde{\varepsilon})\), where \(\tilde{z} \geq 0\) and \(E[-\tilde{\varepsilon}|c_B + \tilde{z}] = 0\). Relabel \(-\tilde{\varepsilon}\) as \(\tilde{\varepsilon}\), and we have proven the first sentence of part 1 of the theorem.

To prove the second sentence of part 1, note that because \(\tilde{c}_A\) and \(\tilde{c}_B\) are monotonically related, \(\tilde{c}_A\) is distributed the same as \(\tilde{c}_B\) only if \(\tilde{c}_A = \tilde{c}_B\). Therefore, if \(\tilde{c}_A \neq \tilde{c}_B\), one or the other of \(\tilde{z}\) or \(\tilde{\varepsilon}\) is not identically zero. Now, if \(\tilde{z}\) is identically zero, then \(\tilde{\varepsilon}\) must not be identically zero, and \(\tilde{c}_A\) is distributed as \(\tilde{c}_B + \tilde{\varepsilon}\), by Jensen’s inequality, we have

\[
E[U_A(\tilde{c}_A)] = E[U_A(\tilde{c}_B + \tilde{\varepsilon})] = E[E[U_A(\tilde{c}_B + \tilde{\varepsilon})|\tilde{c}_B]] < E[U_A(E[\tilde{c}_B|\tilde{c}_B] + E[\tilde{\varepsilon}|\tilde{c}_B])] = E[U_A(\tilde{c}_B)],
\]

which contradicts the optimality of \(\tilde{c}_A\) for agent \(A\). If \(\tilde{\varepsilon}\) is identically zero, then \(\tilde{z}\) must not be, and \(\tilde{c}_A\) is distributed as \(\tilde{c}_B + \tilde{z}\), where \(\tilde{z} \geq 0\) and is not identically zero, therefore, \(\tilde{c}_A\) strictly monotone stochastically dominates \(\tilde{c}_B\), contradicting optimality.
of \( \tilde{c}_B \) for agent \( B \).

We prove part 2 by contradiction. If \( B \) is not weakly more risk averse than \( A \), then there exists a constant \( \hat{c} \), such that \( -\frac{U''_B(c)}{U'(c)} \leq -\frac{U''_A(c)}{U'(A(c))} \). Since \( U_A \) and \( U_B \) are of the class of \( C^2 \), from the continuity of \( -\frac{U''_B(c)}{U'(c)} \), where \( i = A, B \), we get that there exists an interval \( RA \) containing \( \hat{c} \), s.t., \( \forall c \in RA \), \( -\frac{U''_B(c)}{U'(c)} \leq -\frac{U''_A(c)}{U'(A(c))} \). We pick \( c_1, c_2 \in RA \) with \( c_1 < c_2 \). Now we construct agents \( A_1 \) and \( B_1 \), so that \( A_1 \) agrees with \( A \) and \( B_1 \) agrees with \( B \) on \( [c_1, c_2] \), but \( A_1 \) is everywhere more risk averse than \( B_1 \) (and not just on \( RA \)). Specifically, let

\[
U_A(c) = \begin{cases} 
    a_1 - k_1 \exp\left(\frac{U''_A(c_1)}{U'(A(c_1))}c\right) & c < c_1 \\
    U_A(c) & c_1 \leq c \leq c_2 \\
    a_2 - k_2 \exp\left(\frac{U''_A(c_2)}{U'(A(c_2))}c\right) & c > c_2,
\end{cases}
\]

and let

\[
U_{B_1}(c) = \begin{cases} 
    a_3 - k_3 \exp\left(\frac{U''_B(c_1)}{U'(B(c_1))}c\right) & c < c_1 \\
    U_B(c) & c_1 \leq c \leq c_2 \\
    a_4 - k_4 \exp\left(\frac{U''_B(c_2)}{U'(B(c_2))}c\right) & c > c_2,
\end{cases}
\]

where \( k_j \) and \( a_j \) (\( j=1, 2, 3, 4 \)) are determined by the continuity and smoothness of \( U_i(c) \), for \( i = A, B_1 \). For example, \( a_1 \) and \( k_1 \) are determined by:

\[
a_1 - k_1 \exp\left(\frac{U''_A(c_1)}{U'(A(c_1))}c_1\right) = U_A(c_1), \quad \text{and} \quad -k_1 \exp\left(\frac{U''_A(c_1)}{U'(A(c_1))}c_1\right) U''_A(c_1) = U''_A(c_1).
\]

It is easy to see that \( U_{A_1} \) is in the class of \( C^2 \) since from (15), we get:

\[
k_1 = -\left(\frac{U''_A(c_1)}{U'(A(c_1))}\right)^2 \exp\left(\frac{U''_A(c_1)}{U'(A(c_1))}c_1\right) \implies -k_1 \exp\left(\frac{U''_A(c_1)}{U'(A(c_1))}c_1\right) \left(\frac{U''_A(c_1)}{U'(A(c_1))}\right)^2 = U''_A(c_1).
\]

Similarly, we can show that \( U_{B_1}(c) \) is also in the class of \( C^2 \). \( U''_{A_1}(c) < 0, U''_{B_1}(c) < 0 \), and \( -\frac{U''_{B_1}(c)}{U'(B_1(c))} < -\frac{U''_{A_1}(c)}{U'(A_1(c))}, \) for \( \forall c \), i.e., agent \( A_1 \) is more risk averse than \( B_1 \). Fix any \( \lambda_B > 0 \) and choose any random \( \hat{\rho} \) that takes on all the values on \( \left[\frac{U''_{B_1}(c_1)}{U'(B_1(c))}, \frac{U''_{A_1}(c_1)}{U'(A_1(c))}\right] \), then the corresponding \( \tilde{c}_B \) solving the F.O.C. \( U''_{B_1}(\hat{\rho}) = \lambda_B \tilde{c}_B \) takes on all the values on \( [c_1, c_2] \). Because \( U''_B < 0 \), the F.O.C solution is also sufficient, \( \tilde{c}_B \) solves the portfolio
problem for utility function $U_B$, state price density $\tilde{\rho}$ and initial wealth $w_0 = E[\tilde{\rho}c_B]$. Since $U_{B_1} = U_B$ on the support of $\tilde{c}_B$, letting $\tilde{c}_{B_1} = \tilde{c}_B$; $\tilde{c}_{B_1}$ solves the corresponding optimization for $U_{B_1}$ for $\lambda_{B_1} = \lambda_B$.

We now show that there exists $\lambda_{A_1}$ such that $\tilde{c}_{A_1} \equiv I_{A_1}(\lambda_{A_1}\tilde{\rho})$ satisfies the budget constraint $E[\tilde{\rho}c_{A_1}] = w_0$. Due to the choice of $U_{A_1}$, $I_{A_1}(\lambda_{A_1}\tilde{\rho})$ exists and is a bounded random variable for all $\lambda_{A_1}$. Let $\rho = \frac{U'_{A_1}(c_1)}{\lambda_B}$ and $\tilde{\rho} = \frac{U'_{B_1}(c_1)}{\lambda_B}$, then $\tilde{\rho} \in [\underline{\rho}, \overline{\rho}]$, we define $\lambda_1 = \frac{U'_{A_1}(c_1)}{\rho}$ and $\lambda_2 = \frac{U'_{B_1}(c_2)}{\tilde{\rho}}$, i.e., $c_1 = I_{A_1}(\lambda_1\tilde{\rho}) > I_{A_1}(\lambda_1\rho)$ and $c_2 = I_{A_1}(\lambda_2\rho) < I_{A_1}(\lambda_2\tilde{\rho})$. Then, we have $E[\tilde{\rho}I_{A_1}(\lambda_1\tilde{\rho})] < E[\tilde{\rho}c_1] < E[\tilde{\rho}c_B] = w_0$, and $E[\tilde{\rho}I_{A_1}(\lambda_2\tilde{\rho})] > E[\tilde{\rho}c_2] > E[\tilde{\rho}c_B] = w_0$. Since $I_{A_1}(\lambda\tilde{\rho})$ is continuous from the assumption that $U_{A_1}$ is in the class of $C^2$ and $U''_{A_1} < 0$. By the intermediate value theorem, there exists $\lambda_{A_1}$ such that $E[\tilde{\rho}I_{A_1}(\lambda_{A_1}\tilde{\rho})] = w_0$, i.e., $\tilde{c}_{A_1}$ satisfies the budget constraint for $\tilde{\rho}$ and $w_0$.

Now, let us show that $\tilde{c}_{B_1}$ has a wider range of support than that of $\tilde{c}_{A_1}$ when $\tilde{c}_{B_1} \neq \tilde{c}_{A_1}$. Since $A_1$ is more risk averse than $B_1$, from Lemma 1, we know that there exists $c^*$, such that $\tilde{c}_{B_1} \geq \tilde{c}_{A_1}$ when $\tilde{c}_{A_1} \geq c^*$, and $\tilde{c}_{B_1} \leq \tilde{c}_{A_1}$ when $\tilde{c}_{A_1} \leq c^*$. And we can choose $c^* \in [c_1, c_2]$, or else either $\tilde{c}_{B_1} < \tilde{c}_{A_1}$ or $\tilde{c}_{B_1} > \tilde{c}_{A_1}$ and both could not satisfy the budget constraint. Therefore, $\tilde{c}_{B_1}$ has a wider range of support than that of $\tilde{c}_{A_1}$. Let the support of $A_1$’s optimal consumption be $[c_3, c_4] \subseteq [c_1, c_2]$. From the construction of $U_{A_1}$, since $U_{A_1} = U_A$ on the support of $\tilde{c}_{A_1}$, letting $\tilde{c}_A = \tilde{c}_{A_1}$, $\tilde{c}_A$ solves the corresponding optimization for $U_A$ for $\lambda_A = \lambda_{A_1}$.

From the first result of this theorem, we get $\tilde{c}_{B_1} \sim \tilde{c}_{A_1} + \tilde{z}_1 + \tilde{\varepsilon}_1$, where $\tilde{z}_1 \geq 0$ and $E[\tilde{z}_1c_{A_1} + z_1] = 0$. Furthermore, if $\tilde{c}_{A_1} \neq \tilde{c}_{B_1}$, then neither $\tilde{z}_1$ nor $\tilde{\varepsilon}_1$ is identically zero.

We now show that $\tilde{c}_{A_1} \neq \tilde{c}_{B_1}$. If $\tilde{c}_{A_1} = \tilde{c}_{B_1}$, then we pick any two points, for example, $c_3, c_4$ ($c_3 < c_4 \in [c_1, c_2]$) at which both $\tilde{c}_{A_1}$ and $\tilde{c}_{B_1}$ can attain. From the first order conditions, we get: $\frac{U'_{A_1}(c_3)}{U'_{B_1}(c_3)} = \frac{U'_{A_1}(c_4)}{U'_{B_1}(c_4)}$, i.e., $\frac{U''_{A_1}(c_3)}{U''_{B_1}(c_3)} = \frac{U''_{A_1}(c_4)}{U''_{B_1}(c_4)}$. However, from $\frac{U_{A_1}(c)}{U_{B_1}(c)} < -\frac{U''_{A_1}(c)}{U''_{B_1}(c)}$, we have: $\frac{d}{dc}\left(\frac{U'_{A_1}(c)}{U'_{B_1}(c)}\right) < 0$, i.e., $\frac{U''_{A_1}(c)}{U''_{B_1}(c)}$ decreases in $c$. We have: $\frac{U'_{A_1}(c_3)}{U'_{B_1}(c_3)} > \frac{U'_{A_1}(c_4)}{U'_{B_1}(c_4)}$. Contradiction! So, $\tilde{c}_{A_1} \neq \tilde{c}_{B_1}$, we get that neither $\tilde{z}_1$ nor $\tilde{\varepsilon}_1$ is identically zero. Therefore, $E[\tilde{c}_{B_1}] > E[\tilde{c}_{A_1}]$, i.e. $E[\tilde{c}_{B}] > E[\tilde{c}_{A}]$, this contradicts the assumption that $\tilde{c}_A$ is distributed as $\tilde{c}_B + \tilde{z} + \tilde{\varepsilon}$, where $\tilde{z} \geq 0$ and $E[\tilde{\varepsilon}c_B + z] = 0$.

Q.E.D.
Theorem 2 shows that if $B$ is weakly more risk averse than $A$, then $\tilde{c}_A$ is distributed as $\tilde{c}_B$ plus a risk premium plus random noise. Except that the risk premium has mean equal to the difference of the mean consumptions, the distributions of the risk premium and the noise term are typically not uniquely determined. Also, it is possible that the less risk averse agent’s payoff can have a higher mean and a lower variance than the more risk averse agent’s payoff as we have seen in example II.2. This can happen because although adding condition-mean-zero noise always increases variance, adding the non-negative random variable decreases variance if it is sufficiently negatively correlated with the rest. More specifically, we have:

**Corollary 1** If $B$ is weakly more risk averse than $A$, then $\text{Var}(\tilde{c}_A) < \text{Var}(\tilde{c}_B)$ when $\text{Cov}(\tilde{c}_B, \tilde{z}) < -\frac{1}{2} (\text{Var}(\tilde{z}) + \text{Var}(\tilde{\varepsilon}))$.

Proof: We first recall that if $E[\tilde{\varepsilon}|c_B + z] = 0$, then $\text{Cov}(\tilde{\varepsilon}, \tilde{c}_B + \tilde{z}) = 0$. Therefore, $\text{Var}(\tilde{c}_A) = \text{Var}(\tilde{c}_B) + \text{Var}(\tilde{\varepsilon}) + \text{Var}(\tilde{z}) + 2\text{Cov}(\tilde{c}_B, \tilde{z})$. If $\text{Cov}(\tilde{c}_B, \tilde{z}) < -\frac{1}{2} (\text{Var}(\tilde{z}) + \text{Var}(\tilde{\varepsilon}))$, then $\text{Var}(\tilde{c}_A) < \text{Var}(\tilde{c}_B)$.

As it is already known in the literature, Corollary 1 implies that variance may not be a good measure of risk for von Neumann-Morgenstern utility functions, and for general distributions in a complete market, mean-variance preferences are hard to justify. If von Neumann-Morgenstern utility functions are mean-variance preferences, then they have to be quadratic utility functions, but quadratic preferences are not increasing everywhere and they have increasing risk aversion.

Our second main result says that when either of the two agents has non-increasing absolute risk aversion, we can choose $\tilde{z}$ to be non-stochastic.

**Theorem 3** If $B$ is weakly more risk averse than $A$ and either of the two agents has non-increasing absolute risk aversion, then $\tilde{c}_A$ is distributed as $\tilde{c}_B + z + \tilde{\varepsilon}$, where $z = E[\tilde{c}_A - \tilde{c}_B] \geq 0$ and $E[\tilde{\varepsilon}|c_B] = 0$.

Proof: We first consider the case when $A$ has non-increasing absolute risk aversion. Define the utility function $U_{A^*}(\tilde{c}) = U_A(\tilde{c} + E[\tilde{c}_A - \tilde{c}_B])$. Agent $A^*$’s problem with this utility function and wealth $w_0 - E[\tilde{\rho}]E[\tilde{c}_A - \tilde{c}_B]$ is

$$\max_{\tilde{c}} E[U_A(\tilde{c} + E[\tilde{c}_A - \tilde{c}_B])]$$
\begin{equation}
\text{s.t. } E[\hat{\rho}(\hat{c} + E[\hat{c}_A - \hat{c}_B])] = w_0. \tag{16}
\end{equation}

The first order conditions are the same as for optimality of \( \hat{c}_A \) for agent \( A \). To satisfy the budget constraints, agent \( A^* \) will optimally hold \( \hat{c}_A - E[\hat{c}_A - \hat{c}_B] \). Also, \( A^* \) is less risk averse than \( B \) because \( A \) is less risk averse than \( B \) and non-increasing risk aversion of \( A \) implies that \( A^* \) is less risk averse than \( A \). Therefore, by similar arguments as in the proofs of the first result in Lemma 1 and Theorem 2,\(^5\) we can conclude that

\[ \hat{c}_A - E[\hat{c}_A - \hat{c}_B] \sim \hat{c}_B + \bar{z} + \bar{\epsilon}, \tag{17} \]

where \( \bar{z} \geq 0 \), and \( E[\bar{z}|c_B + z] = 0 \). From (17), we have

\[ E[\bar{c}_A] = E[E[\bar{c}_A|c_B + z]] = E[E[\bar{c}_B + E[\bar{c}_A - \bar{c}_B] + \bar{z} + \bar{\epsilon}]|c_B + z] = E[\bar{c}_A] + E[\bar{z}], \]

so \( E[\bar{z}] = 0 \). However, \( \bar{z} \geq 0 \), so we must have \( \bar{z} = 0 \). It follows that

\[ \hat{c}_A - E[\hat{c}_A - \hat{c}_B] \sim \hat{c}_B + \bar{\epsilon}, \text{ i.e., } \hat{c}_A \sim \hat{c}_B + E[\hat{c}_A - \hat{c}_B] + \bar{\epsilon}, \]

where \( E[\bar{\epsilon}|c_B] = 0 \).

In the case that \( B \) has non-increasing absolute risk aversion, the proof is similar except that we define a utility function \( U_B^*(\hat{c}) = U_B(\hat{c} - E[\hat{c}_A - \hat{c}_B]) \), then agent \( B^* \) with this utility function will optimally hold \( \hat{c}_B + E[\hat{c}_A - \hat{c}_B] \) when given initial wealth \( w_0 + E[\hat{\rho}]E[\hat{c}_A - \hat{c}_B] \). Agent \( B^* \) is weakly more risk averse than \( B \) and is therefore more risk averse than \( A \). \( Q.E.D. \)

The non-increasing absolute risk aversion condition is sufficient but not necessary. A quadratic utility function has increasing absolute risk aversion. But, as we illustrate in example V.2, the non-negative random variable can still be chosen to be a constant for quadratic utility functions. If the non-negative random variable can be chosen to be a constant, then we have the following Corollary:

**Corollary 2** If \( B \) is weakly more risk averse than \( A \) and either of the two agents has

\(^5\)The details are different from those in Lemma 1 and Theorem 2 because agents \( A^* \) and \( B \) have different initial wealth. This is also why we can have \( \bar{z} = 0 \) but not \( \bar{\epsilon} = 0 \).
non-increasing absolute risk aversion, then \( \text{Var}(\tilde{c}_A) \geq \text{Var}(\tilde{c}_B) \).

Proof: If the non-negative random variable can be chosen to be a constant, then we have \( E(\tilde{\varepsilon}|\tilde{c}_B) = 0 \), i.e., \( \text{Cov}(\tilde{\varepsilon}, \tilde{c}_B) = 0 \). Therefore, \( \text{Var}(\tilde{c}_A) = \text{Var}(\tilde{c}_B) + \text{Var}(\tilde{\varepsilon}) + 2\text{Cov}(\tilde{c}_B, \tilde{\varepsilon}) = \text{Var}(\tilde{c}_B) + \text{Var}(\tilde{\varepsilon}) \geq \text{Var}(\tilde{c}_B) \).

IV. Extension to a Multiple-Period Model

We now examine our main results in a multiple period model. We assume that each agent’s problem is:

**Problem 2**

\[
\max_{\tilde{c}_t} E\left[ \sum_{t=1}^{T} D_t U(\tilde{c}_t) \right],
\]

s.t. \( E\left[ \sum_{t=1}^{T} \tilde{\rho}_t \tilde{c}_t \right] = w_0 \), \hspace{1cm} (18)

where \( D_t \) is a discount factor (e.g., \( D_t = e^{-\kappa t} \) if the pure rate of time discount \( \kappa \) is constant), \( U(\cdot) \) is the utility function \( U_A(\cdot) \) or \( U_B(\cdot) \), and \( \tilde{\rho}_t \) is the state price density in period \( t \). Again, we will assume that both agents have optimal random consumptions, called \( \tilde{c}_{At} \) and \( \tilde{c}_{Bt} \). The first order condition gives us \( U'(\tilde{c}_t) = \lambda \frac{\tilde{\rho}_t}{D_t} \). Therefore, the first order conditions for the two agents’ optima are given by

\[
U'_A(\tilde{c}_{At}) = \lambda_A \frac{\tilde{\rho}_t}{D_t}, \quad U'_B(\tilde{c}_{Bt}) = \lambda_B \frac{\tilde{\rho}_t}{D_t},
\]

i.e.,

\[
\tilde{c}_{At} = I_A \left( \lambda_A \frac{\tilde{\rho}_t}{D_t} \right), \quad \tilde{c}_{Bt} = I_B \left( \lambda_B \frac{\tilde{\rho}_t}{D_t} \right),
\]

where \( I_A \) and \( I_B \) are the inverse function of \( U'_A(\cdot) \) and \( U'_B(\cdot) \) respectively, by negativity of the second derivatives, \( \tilde{c}_{At} \) and \( \tilde{c}_{Bt} \) are decreasing functions of \( \tilde{\rho}_t \). By similar arguments in the one period model, we will have the following results

**Lemma 2** If \( B \) is more risk averse than \( A \), then
1. there exists some critical consumption level \( c^*_t \) (can be \( \pm \infty \)) such that \( \tilde{c}_{At} \geq \tilde{c}_{Bt} \) when \( \tilde{c}_{At} \geq c^*_t \), and such that \( \tilde{c}_{At} \leq \tilde{c}_{Bt} \) when \( \tilde{c}_{At} \leq c^*_t \);

2. if it happens that the budget shares as a function of time are the same for both agents at some time \( t \), i.e., \( E[\tilde{\rho} t \tilde{c}_{At}] = E[\tilde{\rho} t \tilde{c}_{Bt}] \), then \( E[\tilde{c}_{At}] \geq E[\tilde{c}_{Bt}] \), and we have \( \tilde{c}_{At} \sim \tilde{c}_{Bt} + \tilde{z}_t + \tilde{\epsilon}_t \), where \( \tilde{z}_t \geq 0 \) and \( E[\tilde{\epsilon}_t | \tilde{c}_{Bt} + \tilde{z}_t] = 0 \). And if \( \tilde{c}_{At} \neq \tilde{c}_{Bt} \), then neither \( \tilde{z}_t \) nor \( \tilde{\epsilon}_t \) is identically zero. In particular, if the budget shares are the same for all \( t \), then this distributional condition holds for all \( t \).

The proof of Lemma 2 is the same as that in the one-period model (Lemma 1, and Theorem 2, part 1). If the \( D_t \)’s are not the same for both agents, or the same for the two agents without any restriction on budget shares, then the distributional condition need not hold in any period. For example, if the more risk averse agent \( B \) spends most of the money earlier but the less risk averse agent \( A \) spends more later, then the mean payoff could be higher in an earlier period for the more risk averse agent, i.e., \( E[\tilde{c}_{Bt}] > E[\tilde{c}_{At}] \).

Now, assume both agents have the same discount factor \( D_t \) and choose the period and consumption using a mixture model: first choose \( t \) with probability \( \mu_t = \frac{D_t}{\sum_{t=1}^{T} D_t} \), and then choose \( \rho_t \) from its distribution. Then, we will show that, under this probability measure \( \tilde{c}_A \sim \tilde{c}_B + \tilde{z} + \tilde{\epsilon} \).

**Definition 1** Suppose the original probability space has probability measure \( P \) over states \( \Omega \) with filtration \( \{\mathcal{F}_t\} \). Define the discrete random variable \( \tau \) on associated probability space \( (\Omega^*, \mathcal{F}^*, P^*) \) so that \( P^*(\tau = t) = \mu_t = \frac{D_t}{\sum_{t=1}^{T} D_t} \). Define the state of nature in the product space \( (t, \omega) \in \hat{\Omega} \equiv \Omega^* \times \Omega \) with \( t \) and \( \omega \) drawn independently. The \( \sigma \)-algebra \( \hat{\mathcal{F}} \) is the completion of \( \mathcal{F}^* \times \mathcal{F}_\tau \). The synthetic probability measure is the one consistent with independence generated from \( \hat{P}(f^*, f) = P^*(f^*) \times P(f) \) for all subsets \( f^* \in \mathcal{F}^* \) and subsets \( f \in \mathcal{F}_\tau \).

The synthetic probability measure assigns a probability measure that looks like mixture model, drawing time first assigning probability \( \mu_t \) to time \( t \), and then drawing from \( \rho_t \) using its distribution in the original problem.

Recall that under the original probability measure, each agent’s problem is given in (18). Now we want to write down an equivalent problem, in terms of the choice
of distribution of each $\tilde{c}_t$, but with the new synthetic probability measure. The consumption $\tilde{c}$ under the new probability space over which synthetic probabilities are defined is a function of $\rho$ and $t$; we identify $\tilde{c}(\rho, t)$ with what used to be $\tilde{c}_t(\rho)$. To write the objective function in terms of the synthetic probabilities, we can write

$$E\left[\sum_{t=1}^{T} D_t U(\tilde{c}_t)\right] = \sum_{t=1}^{T} D_t E[U(\tilde{c}_t)] = \sum_{t=1}^{T} \left(\sum_{s=1}^{T} D_s\right) \mu_t \hat{E}[U(\tilde{c})|t]$$

$$= \left(\sum_{s=1}^{T} D_s\right) \sum_{t=1}^{T} \mu_t \hat{E}[U(\tilde{c})|t] = \left(\sum_{s=1}^{T} D_s\right) \hat{E}[U(\tilde{c})],$$

(19)

where $\hat{E}$ denotes the expectation under the synthetic probability. $\sum_{s=1}^{T} D_s$ is a positive constant, so the objective function is equivalent to maximizing $\hat{E}[U(\tilde{c})]$.

Now, we can write the budget constraint in terms of the synthetic probabilities,

$$w_0 = E[\sum_{t=1}^{T} \tilde{\rho}_t \tilde{c}_t] = \sum_{t=1}^{T} \mu_t E[\frac{\tilde{\rho}_t}{\mu_t} \tilde{c}_t] = \sum_{t=1}^{T} \mu_t \hat{E}[\frac{\rho}{\mu} \tilde{c}|t] = \hat{E}[\frac{\rho}{\mu} \tilde{c}].$$

(20)

Then we can apply our single-period result (Theorem 2) to derive that the distributional condition holds on a mixture model of the $\tilde{c}_A$ and $\tilde{c}_B$ over time:

**Theorem 4** In a multiple-period model, let $\tilde{c}_A$ and $\tilde{c}_B$ be the optimal consumption of $A$ and $B$ respectively under the synthetic probability measure, if $B$ is more risk averse than $A$, then, $\tilde{c}_A \sim \tilde{c}_B + \tilde{z} + \tilde{\epsilon}$ under the synthetic probabilities, where $\tilde{z} \geq 0$, $\hat{E}[\tilde{\epsilon}|c_B + z] = 0$.

Therefore, if the budget shares are not the same for both agents at each time period $t$, then as we have seen in Lemma 2, the distributional result need not hold period-by-period in a multi-period model with identical weights over time. However, Theorem 4 implies that the distributional condition still holds under the synthetic probabilities in a multi-period model. This result retains the spirit of our main result while acknowledging that changing risk aversion may cause consumption to shift over time.
V. More Examples

In this section, we illustrate our main results using negative exponential, power, quadratic and hyperbolic absolute risk aversion (HARA) utility functions. First we have a result that is useful for analyzing the examples.

Lemma 3 For any random variable \( x \), and all \( m > 1 \), \( x - E[x] \) second order stochastically dominates \( m(x - E[x]) \).

Proof: Let \( x_1 = x - E[x], \ x_2 = m(x - E[x]) \), we have

\[
F_{x_2}(q) = \text{Prob}[m(x - E(x)) < q] = \text{Prob}[x - E(x) < \frac{q}{m}] = F_{x_1}(\frac{q}{m}),
\]

therefore,

\[
\begin{cases}
F_{x_2}(q) \leq F_{x_1}(q) & q \geq 0 \\
F_{x_2}(q) > F_{x_1}(q) & q < 0.
\end{cases}
\]

Because \( \int_{-\infty}^{+\infty} (F_{x_2}(q) - F_{x_1}(q))dq = 0 \), we have \( S(z) = \int_{-\infty}^{z} (F_{x_2}(q) - F_{x_1}(q))dq \geq 0, \forall z \), i.e., \( x - E[x] \) second order stochastically dominates \( m(x - E[x]) \). Q.E.D.

Example V.1 (Negative Exponential Utility (CARA)) B is more risk averse than A, A and B have the same initial wealth \( w_0 \) and the utility functions are as follows

\[
U_A(c) = -e^{-a_1c}, \quad U_B(c) = -e^{-a_2c}, \quad a_1 < a_2.
\]

The first order conditions give us

\[
\frac{U_A'(\tilde{c}_A)}{U_B'(\tilde{c}_B)} = k, \quad k > 0. \quad i.e., \quad a_1 \tilde{c}_A = a_2 \tilde{c}_B - \ln k,
\]

from (21), we have

\[
E[\rho a_1 \tilde{c}_A] = E[\rho a_2 \tilde{c}_B] - E[\rho \ln k], \quad i.e., \quad \ln k = -\frac{a_1 - a_2}{E[\rho]} w_0.
\]

Therefore,

\[
\tilde{c}_A - E[\rho]^{-1} w_0 = \frac{a_2}{a_1} (\tilde{c}_B - E[\rho]^{-1} w_0),
\]

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it follows that
\[ \tilde{c}_A - E[\tilde{c}_A] = \frac{a_2}{a_1} (\tilde{c}_B - E[\tilde{c}_B]). \]
From Lemma 3, we get \( \tilde{c}_B - E[\tilde{c}_B] \) second stochastic dominates \( \tilde{c}_A - E[\tilde{c}_A] \). It follows that \( \tilde{c}_A - E[\tilde{c}_A] \sim \tilde{c}_B - E[\tilde{c}_B] + \varepsilon \), i.e., \( \tilde{c}_A \sim \tilde{c}_B + E[\tilde{c}_A] - E[\tilde{c}_B] + \varepsilon \), and \( E[\varepsilon | c_B] = 0 \).

**Example V.2 (Quadratic Utility (IARA))** \( B \) is more risk averse than \( A \), and \( A \) and \( B \) have the same initial wealth \( w_0 \) and the utility functions are as follows
\[
U_A(c) = -\frac{1}{2} \left( \frac{1}{a_1} - c \right)^2, \quad U_B(c) = -\frac{1}{2} \left( \frac{1}{a_2} - c \right)^2, \quad \frac{1}{a_i} - c > 0, i = 1, 2, a_1 < a_2. 
\]
The first order conditions give us
\[
\frac{U_A'(\tilde{c}_A)}{U_B'(\tilde{c}_B)} = k, \quad k > 0. \quad \text{i.e.,} \quad \frac{1}{a_1} - \tilde{c}_A = k \left( \frac{1}{a_2} - \tilde{c}_B \right), \quad (22)
\]
From (22), we have
\[
k = \frac{1 - a_2 E[\rho] - w_0}{1 - a_1 E[\rho] - w_0} > 1.
\]
Therefore, \( \tilde{c}_A - E[\tilde{c}_A] = k(\tilde{c}_B - E[\tilde{c}_B]) \) with \( k > 1 \), from Lemma 3, we have \( \tilde{c}_A - E[\tilde{c}_A] \sim \tilde{c}_B - E[\tilde{c}_B] + \varepsilon \), i.e., \( \tilde{c}_A \sim \tilde{c}_B + E[\tilde{c}_A] - E[\tilde{c}_B] + \varepsilon \), and \( E[\varepsilon | c_B] = 0 \).

**Example V.3 (Power Utility (DARA))** \( B \) is more risk averse than \( A \), and \( A \) and \( B \) have the same initial wealth \( w_0 \) and the utility functions are as follows
\[
U_A(c) = \frac{c^{1-\gamma_1}}{1-\gamma_1}, \quad U_B(c) = \frac{c^{1-\gamma_2}}{1-\gamma_2}, \quad \gamma_i > 0, \quad \gamma_i \neq 1, i = 1, 2.
\]
The first order conditions give us
\[
\tilde{c}_A = k\tilde{c}_B^{\gamma_2/\gamma_1}, \quad k > 0, \quad (23)
\]
it is not hard to see that
\[
\begin{aligned}
\bar{c}_A &\geq \bar{c}_B & \bar{c}_B &\geq (\frac{1}{k})^{\gamma_1/(\gamma_2-\gamma_1)} \\
\bar{c}_A &< \bar{c}_B & \bar{c}_B &< (\frac{1}{k})^{\gamma_1/(\gamma_2-\gamma_1)}.
\end{aligned}
\]
From Theorem 2 and 3, it follows that $\bar{c}_A \sim \bar{c}_B + E[\bar{c}_A] - E[\bar{c}_B] + \varepsilon$, and $E[\varepsilon|c_B] = 0$.

**Example V.4 (Hyperbolic Absolute Risk Aversion (HARA))**

quad B is more risk averse than A, A and B have the same initial wealth $w_0$ and the utility functions are as follows

\[
U_i(c) = \begin{cases} 
\frac{(c-c_{i0})^{1-\gamma}}{1-\gamma} & \gamma > 0, \gamma \neq 1 \\
\log(c-c_{i0}) & \gamma \neq 1,
\end{cases}
\]

where $c_{i0}(i = A, B)$ is the subsistence consumption for agent A and B, $c_{A0} < c_{B0}$.

The first order conditions give us

\[
\bar{c}_A - c_{A0} = k(\bar{c}_B - c_{B0}), \quad (24)
\]

From (24), we get

\[
k = \frac{w_0 - c_{A0}E[\rho]}{w_0 - c_{B0}E[\rho]} > 1,
\]

from Lemma 3, we have $\bar{c}_A - E[\bar{c}_A] \sim \bar{c}_B - E[\bar{c}_B] + \varepsilon$, i.e., $\bar{c}_A \sim \bar{c}_B + E[\bar{c}_A] - E[\bar{c}_B] + \varepsilon, E[\varepsilon|c_B] = 0$.

The following example provides details for example II.2. It says that the random variable $\bar{z}$ cannot be chosen to be a constant in some case.

**Example V.5** B is more risk averse than A, A and B have the same initial wealth $w_0$ and the utility functions are as follows

\[
U_A(c) = -\left(\frac{1}{a} - c\right)^{n_A}, \quad U_B(c) = -\left(\frac{1}{a} - c\right)^{n_B}, \quad \text{where } \frac{1}{a} - c > 0, \quad n_A < n_B.
\]
From (5), we get:

\[ n_A \left( \frac{1}{a} - \tilde{c}_A \right)^{n_A - 1} = \frac{\lambda_A}{\lambda_B} n_B \left( \frac{1}{a} - \tilde{c}_B \right)^{n_B - 1}, \]

i.e.,

\[ \tilde{c}_A = \frac{1}{a} - \left( \frac{\lambda_A n_B \left( \frac{1}{a} - \tilde{c}_B \right)^{n_B - 1}}{\lambda_B n_A} \right)^{\frac{1}{n_A - 1}}. \]  

(25)

We have:

\[ \begin{cases} 
\tilde{c}_A \geq \tilde{c}_B & \tilde{c}_B \geq \frac{1}{a} - \left( \frac{n_A \lambda_B}{n_B \lambda_A} \right)^{\frac{1}{n_B - n_A}} \\
\tilde{c}_A < \tilde{c}_B & \tilde{c}_B < \frac{1}{a} - \left( \frac{n_A \lambda_B}{n_B \lambda_A} \right)^{\frac{1}{n_B - n_A}}. 
\end{cases} \]

From the proof of Theorem (2), we know that \( \tilde{c}_A \sim \tilde{c}_B + \tilde{z} + \tilde{\varepsilon} \) and \( E[\tilde{z}|\tilde{c}_B + \tilde{z}] = 0 \). However, we can see that \( \tilde{z} \) may not be chosen to be a constant \( E[\tilde{c}_A] - E[\tilde{c}_B] \) from previous discussion in Section II.

VI. Concluding Remarks

In a complete market, we show that an agent who is less risk averse than another will choose a portfolio whose payoff is distributed as the other’s payoff plus a nonnegative random variable plus conditional-mean-zero noise. This result holds for any strictly concave \( C^2 \) utility function. If either agent has non-increasing absolute risk aversion, then the non-negative random variable can be chosen to be a constant. The non-increasing absolute risk aversion condition is sufficient but not necessary. We also provide a counter example, such that, in general, this non-negative random variable cannot be chosen to be a constant.

We further show a converse theorem. If the first agent chooses a payoff that is distributed as the second’s payoff, plus a non-negative random variable, plus conditional-mean-zero noise in all complete markets, then the first agent is less risk averse than the second agent. We also extend our main results to a multiple period model. Due to shifts in the timing of consumption, agents’s optimal consumption at each date may not be ordered when risk aversion changes. However, for agents with the same
pure rate of time preference, there is a weighting of probabilities across periods that preserves the single-period result.

The optimal consumption may not be ordered for agents with different risk aversion when agents’ utility functions are concave but not strictly concave as we have shown in example II.3. Intuitively, the problem is that even with identical preferences, two different optimal consumptions may not be ordered. We conjecture that our main results hold for weakly concave preferences for some canonical choice of optimal consumption for each agent. We leave this to future research. Our paper derives comparative statics results in complete markets for agents with von Neumann-Morgenstern preferences. Machina (1989) has shown that many previous comparative statics results generalize to the broader class of Machina preferences (Machina (1982)). Our proofs do not generalize obviously to this class, but we conjecture that our results are still true for Machina preferences.
References


