DERIVATIVE SECURITIES
Lecture 4: The Black-Scholes Model

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- Black-Scholes option pricing model
- Lognormal price process
- Call price
- Put price
- Using Black-Scholes

Continuous-Time Option Pricing

We have been using the binomial option pricing model of Cox, Ross, and Rubinstein [1979]. In this lecture, we go back to the original modern option pricing model of Black and Scholes [1973]. The mathematical underpinnings of the Black-Scholes model would take a couple of semesters to develop in any formal way, but we can discuss the intuition by viewing it as the limit of the binomial model as the time between trades becomes small.

Black, Fischer, and Myron Scholes (1973) "The Pricing of Options and Corporate Liabilities" *Journal of Political Economy* **81**, 637–654

Cox, John C., Stephen A. Ross, and Mark Rubinstein (1979) "Option Pricing: A Simplified Approach" *Journal of Financial Economics* **7**, 229–263

Lognormal stock price process

The stock price process in the Black-Scholes model is lognormal, that is, given the price at any time, the logarithm of the price at a later time is normally distributed. For $t > s$,

$$\log(S_t/S_s) \sim_d N((\mu - \sigma^2/2)(t-s), \sigma^2(t-s))$$

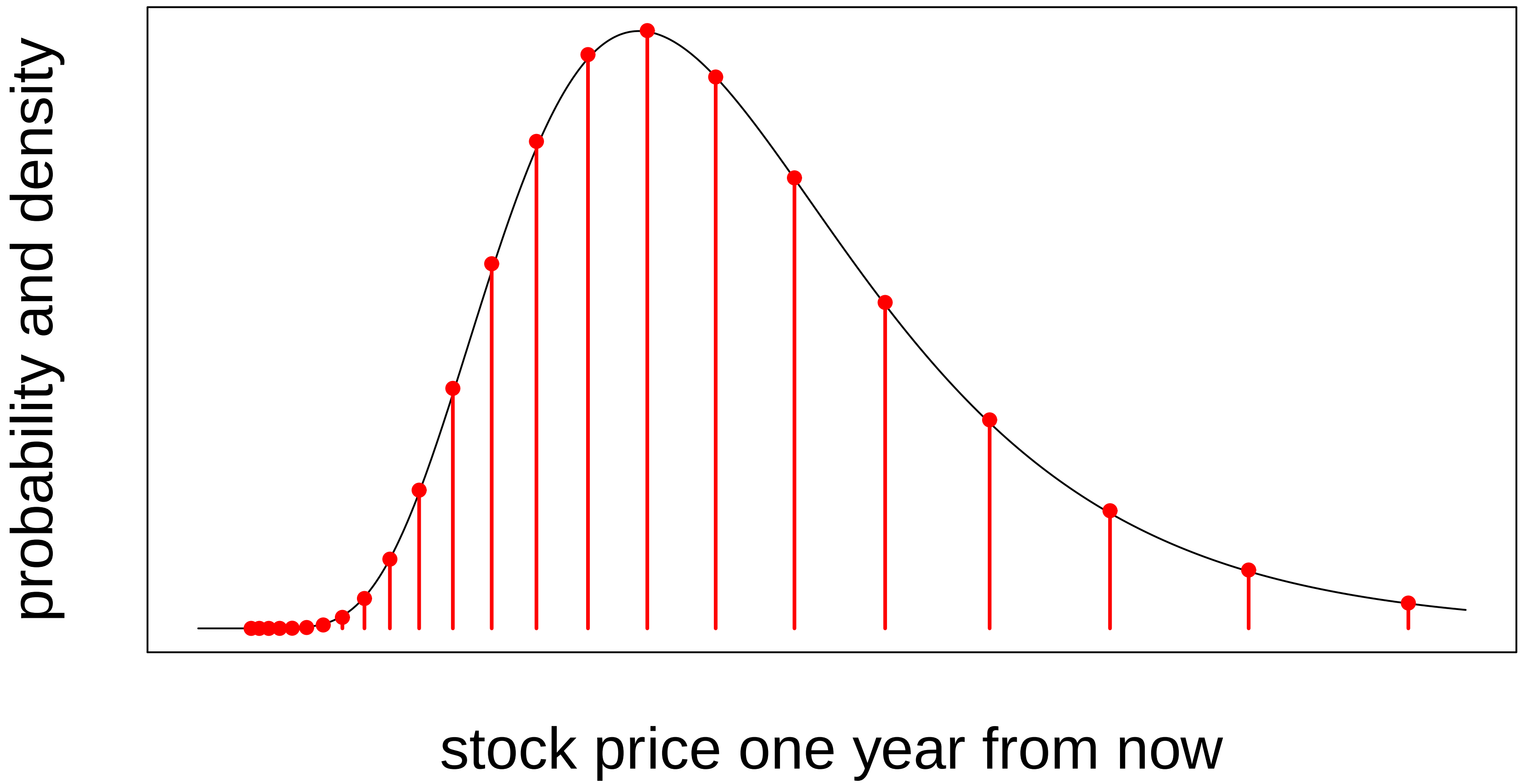
(Recall that the normal distribution is the familiar "bell curve" from statistics.) It is also known how to do option pricing for a continuous-time model with normally distributed prices, but the lognormal model is more reasonable because stocks have limited liability and cannot go negative. In the basic Black-Scholes model there are no dividends. Over a short period of time of length $t-s$, the mean rate of return is approximately $\mu(t-s)$ and the variance is approximately $\sigma^2(t-s)$:

$$E\left[\frac{S_t - S_s}{S_s}\right] = e^{\mu(t-t)} - 1 \approx \mu(t-s)$$

and the variance is approximately

$$\mathrm{var}(\frac{S_t - S_s}{S_s}) \approx \sigma^2(t-s).$$

Binomial approximates lognormal when time increment is small



Black-Scholes call option pricing formula

The Black-Scholes call price is

$$C(S, B, \sigma^2 T) = SN(x_1) - BN(x_2)$$

where $N(\cdot)$ is the unit normal cumulative distribution function,[1] T is the time-to-maturity, σ^2 is the variance per unit time, B is the price $Xe^{-r_f T}$ of a discount bond maturing at T with face value X,

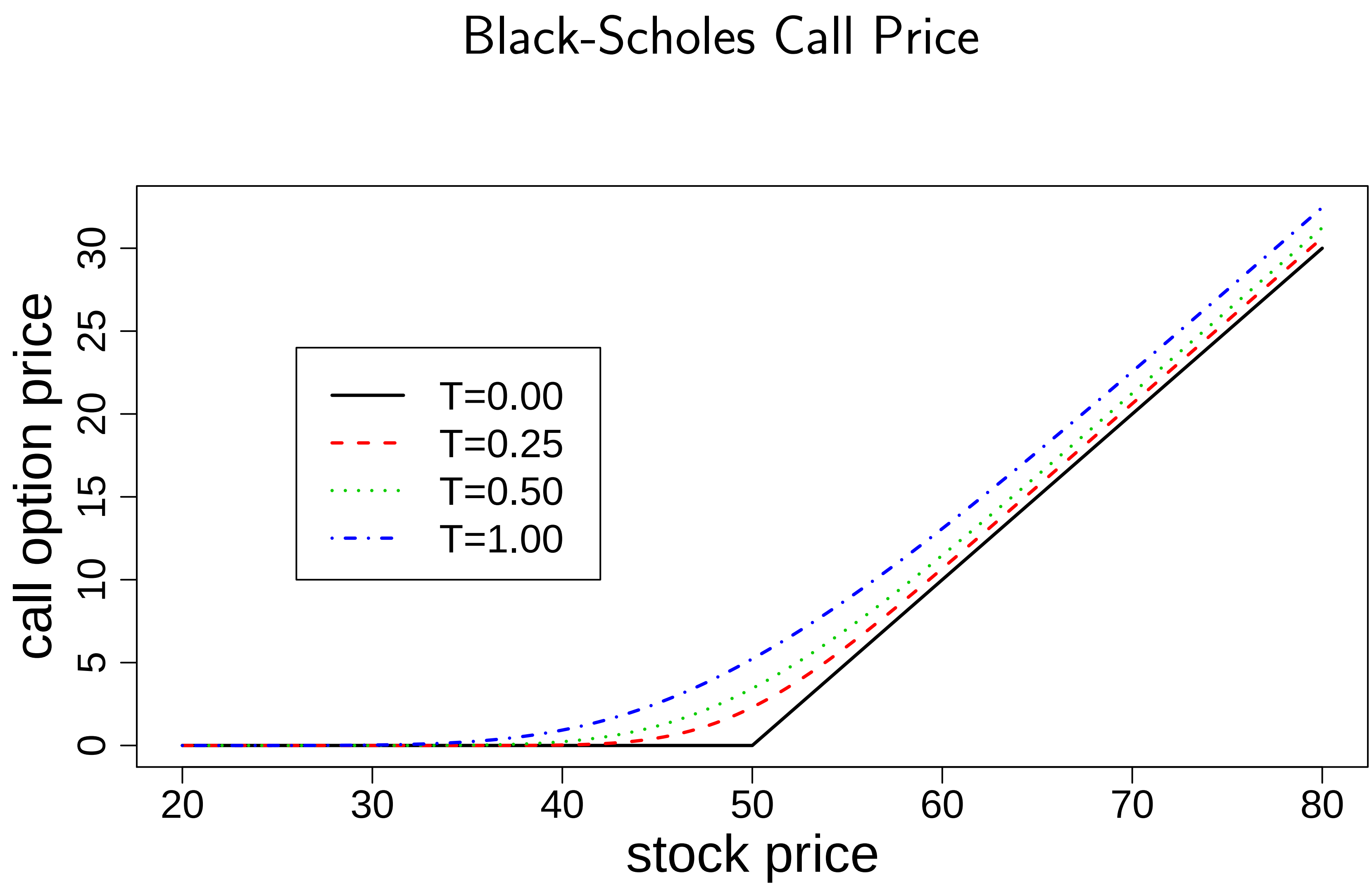
$$x_1 = \frac{\log(S/B)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T},$$

and

$$x_2 = \frac{\log(S/B)}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T}.$$

Note that the Black-Scholes option price does not depend on the mean return of the stock. This is because the risk-neutral probabilities have the same local variance of the stock returns as in actual probabilities, but the mean return in risk-neutral probabilities (as always) equals the riskfree rate. Note that these prices are for *European options* that *pay no dividends*.

[1] Reminder: A unit normal random variable is normally distributed with mean 0 and variance 1. The distribution function of a random variable y is defined by $N(x) = \text{prob}(x \leq y)$ where prob denotes the probability.


Black-Scholes Call Price
call option price
stock price
0
5
10
15
20
25
30
20
30
40
50
60
70
80
T=0.00
T=0.25
T=0.50
T=1.00

Intuition: gambler's rule[2]

When a call option ends in the money at maturity T periods from now, its value is $S_T - X_T$, which has value $S - e^{-r_f T} = S - B$ in the notation above. If the probability of exercise were π and chosen independent of everything in the economy, the option value would be $S\pi - B\pi$. However, we can skew the odds in our favor if we pick and choose when to exercise, which skews the probabilities: $SN(x_1) - BN(x_2)$ where $x_1 > x_2$. As maturity approaches, both probabilities tend to 1 if the option is in the money or 0 if the option is out of the money. In particular, if we look at x_1 and x_2, they both go to $+\infty$ as the moneyness S/B of the option increases or goes down to $-\infty$ as the moneyness decreases. The difference between x_1 and x_2 is the greatest when option is near the money and volatility and time-to-maturity are large.

[2] fanciful description due to Stephen Brown

Where does Black-Scholes come from?

The Black-Scholes formula can be derived as the limit of the binomial pricing formula as the time between trades shrinks, or directly in continuous time using an arbitrage argument. The option value is a function of the stock price and time, and the local movement in the stock price can be computed using a result called Itô's lemma, which is an extension of the chain rule from calculus. The standard version of the chain rule does not work, because the stock price in the lognormal model is not differentiable (and cannot be or else stock price would be locally predictable the existence of an arbitrage). Even if a function of the stock has zero derivative at a point, its expected rate of increase can be positive due to the volatility of local price movements.

Once Itô's lemma is used to calculate the local change in the option value in term of derivatives of the function of stock price and time, absence of arbitrage implies a restriction on the derivatives of the function (in economic terms, risk premium is proportional to risk exposure), essentially similar to the per-period hedge in the binomial model. The absence of arbitrage implies a differential equation that is solved subject to the boundary conditions including the known option value at the end.

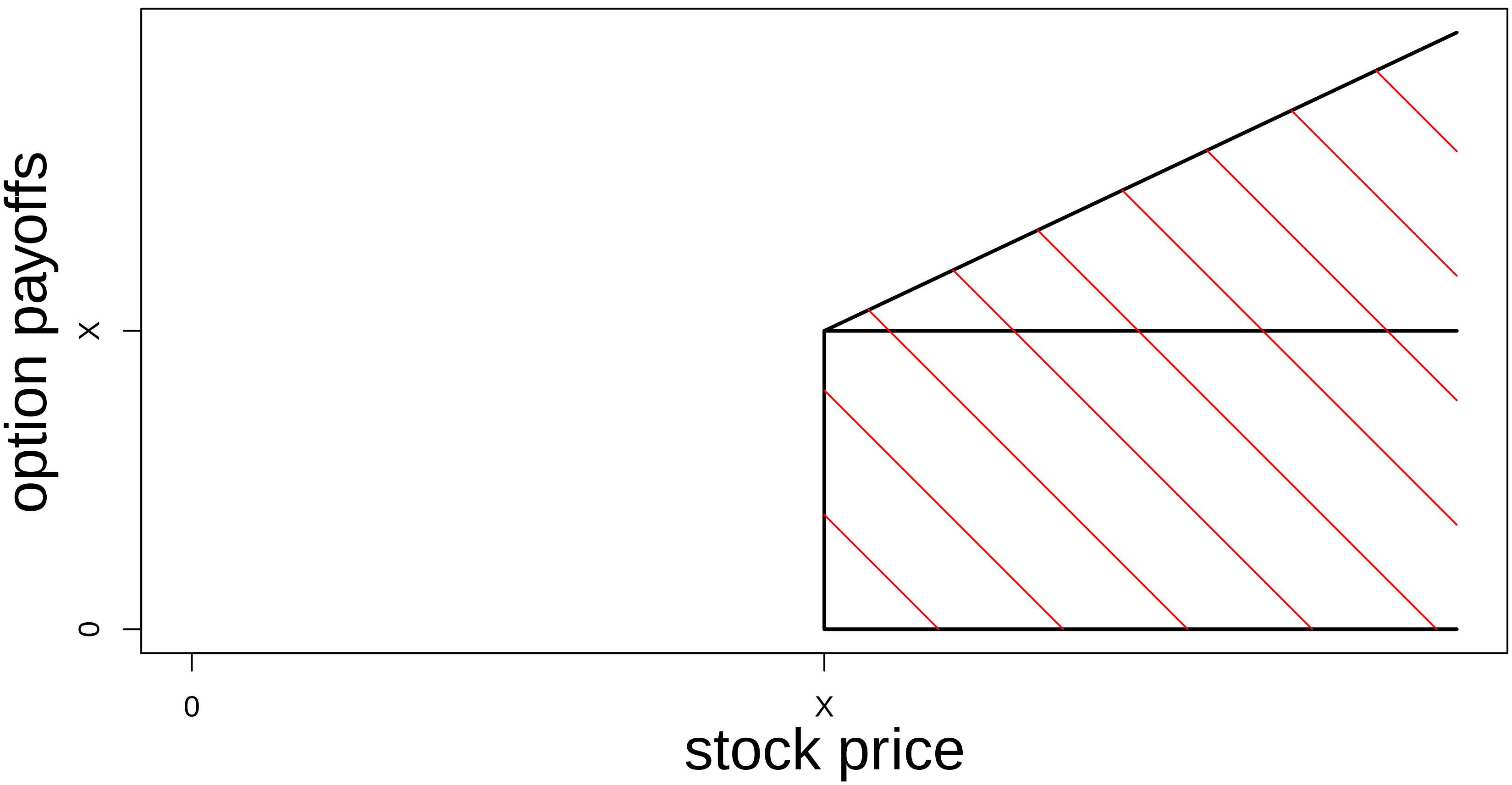
What are the two terms?

I am not sure it widely known, but the two terms in Black-Scholes call formula are prices of digital options. The first term $SN(x_1)$ is the price of a digital option that pays one share of stock at maturity when the stock price exceeds X: this is a digital option if we measure payoff in terms of the stock price (this is called using the stock as numeraire and is like a currency conversion). The second term $XN(x_2)$ is the price of a short position in a digital option that pays X at maturity when the stock price exceeds X. There is a slightly mystical result that the two terms also represent the portfolio we hold to replicate the option if we want to perform an arb. We can create the call option payoff at the end by holding the dynamic portfolio that has at each point of time $SN(x_1)$ shares long in stocks and $BN(x_2)$ short in bonds.[3]

It is also possible to write the Black-Scholes (European) put formula as two terms, each of which is the value of a digital option.

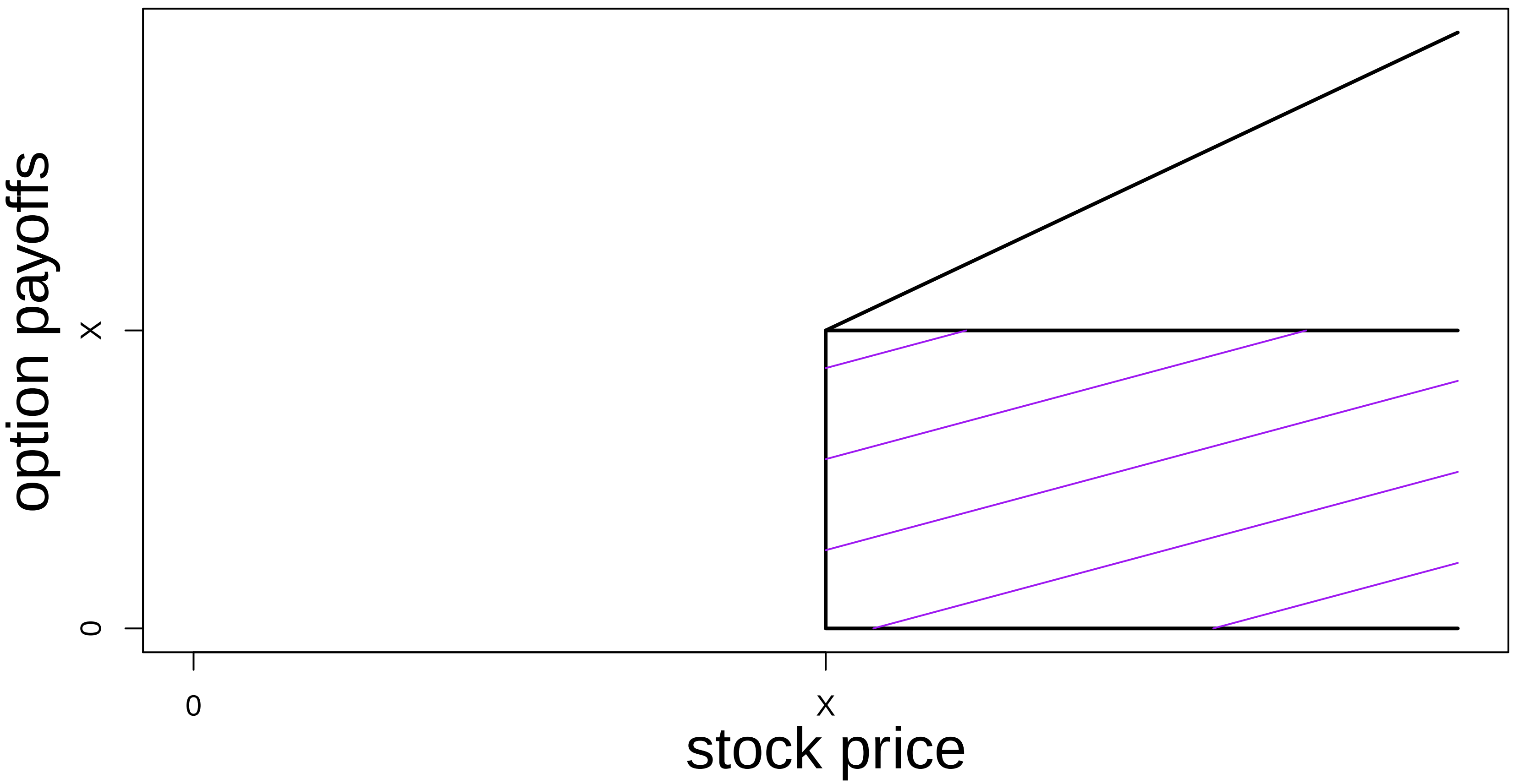
[3]More details for the curious: Let $y = S_T/S_0$, which has the same risk-neutral density $f(y)$ independent of S_0. We can write the call price as $\int_{y=X/S_0}^{\infty}(S_0 y - X)f(y)dy$. Differentiating this with respect to S_0, the part of the derivative which comes from the derivative of the lower endpoint with respect to the limit is zero because the integrand vanishes there. However, this would not be true for either digital option. In terms of the formula, the derivatives corresponding to x_1 and x_2 are not zero but they cancel.

Two terms in Black-Scholes



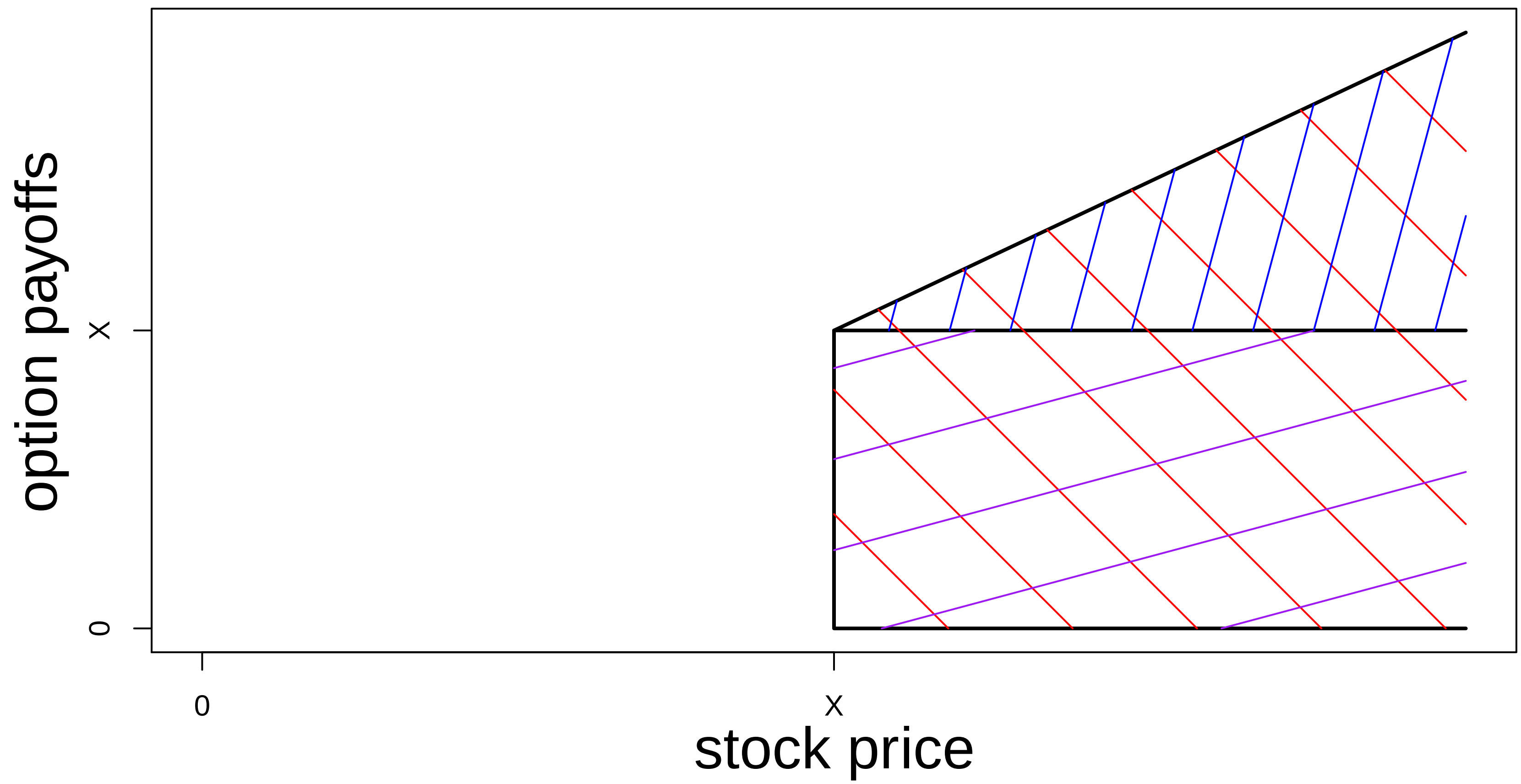
$SN(x_1)$ claim paying the stock price but only when it is larger than X.

Two terms in Black-Scholes



$XN(x_1)$ claim paying the strike price when the stock price is larger than X.

Two terms in Black-Scholes

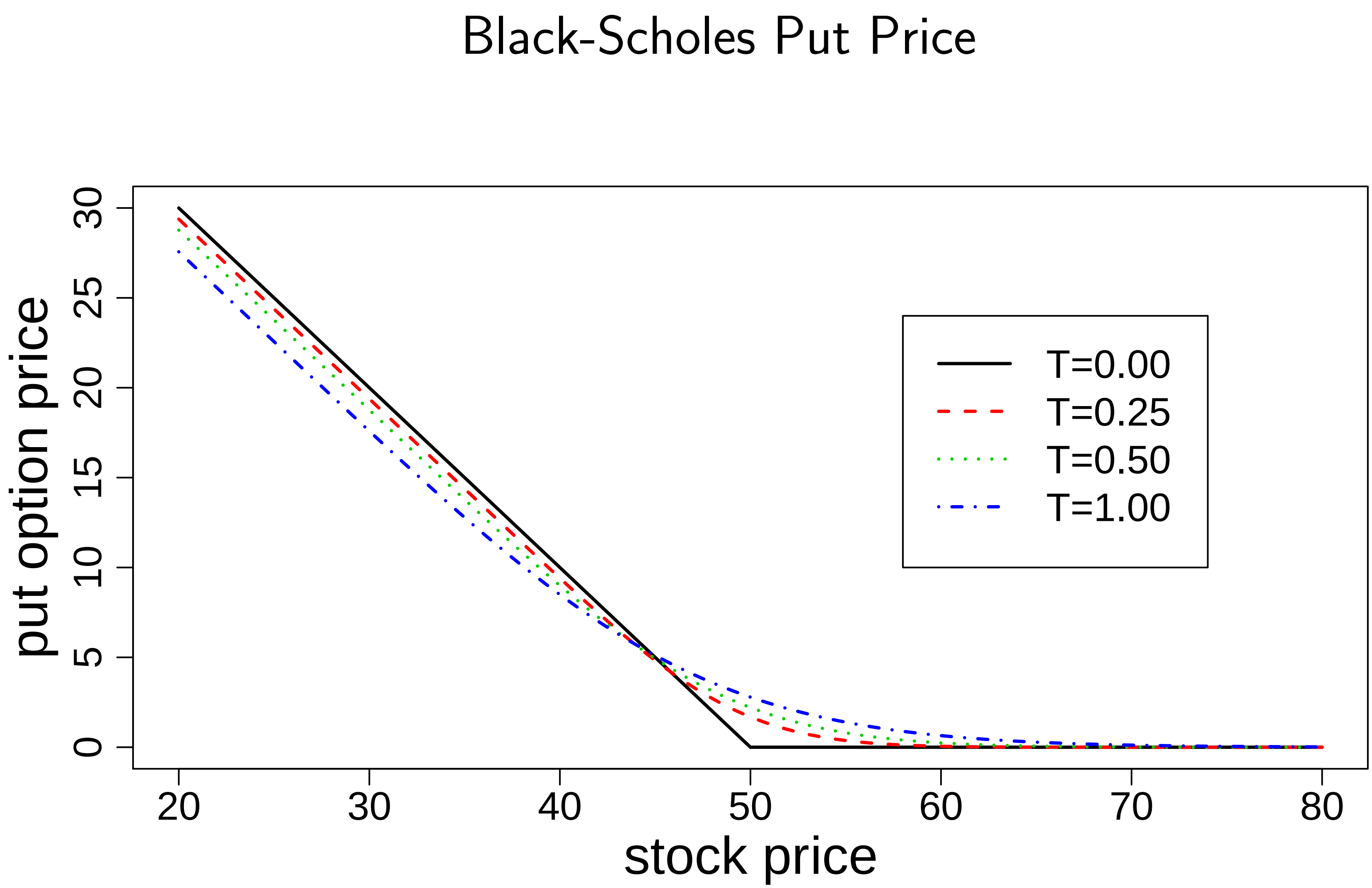


The difference $SN(x_1) - XN(x_2)$ is the Black-Scholes call price.

In-class exercise: Black-Scholes put price

Derive the Black-Scholes put price (for an American option on a stock that is not expected to pay dividends between now and maturity).

hint: Use the known form of the Black-Scholes call price $(SN(x_1) - BN(x_2)$ and put-call parity $(C + B = P + S)$.


Black-Scholes Put Price
put option price
stock price
T=0.00
T=0.25
T=0.50
T=1.00
0
5
10
15
20
25
30
20
30
40
50
60
70
80

Using the Black-Scholes Model

The Black-Scholes model is usually the model of choice when working with a plain vanilla European option pricing application. The binomial model is more flexible and is a better choice for inclusion of a nontrivial American feature, realistic dividends, and other complications. The simplest way to obtain the Black-Scholes call price is to use available tables, a spreadsheet, or a financial calculator. Or, compute the value directly using the formula. The value of $N(\cdot)$ is available in tables or from the approximation in the next slide. Be sure to use the natural logarithm function to compute $\log(S/B)$.

Cumulative normal distribution function $N(\cdot)$

If you want to calculate the Black-Scholes formula yourself, there are lots of good approximations to the cumulative normal distribution function $N(x)$. For example, you can use the following procedure for $x \leq 0$. For $x > 0$, use the procedure on $-x$ and then use the formula $N(x) = 1 - N(-x)$. First compute t, defined by $t \equiv 1/(1 - 0.33267x)$. Then, use t and x to compute $N(x) \approx (0.4361836t - 0.1201676t^2 + 0.937298t^3)exp(-x^2/2)/\pi$, where $\pi \approx 3.1415926535$ is the familiar numerical constant.[4]

If x is very close to zero (say $-.25 < x < .25$), as it will be when computing the Black-Scholes price for near-the-money options at short maturities, $N(x) \approx .5 + x/\sqrt{2\pi} \approx .5 + 0.39894x$. Consequently, near-the-money call options are worth about $\frac{S-B}{2} + .4\frac{S+B}{2}\sigma\sqrt{T}$ This is a handy approximation for a meeting or job interview where a quick approximate answer is useful.

[4]This formula and other approximations are available in Abramowitz and Stegun, *Handbook of Mathematical Functions*, a famous math book available online at http://people.math.sfu.ca/ cbm/aands/

What interest rate to use

The original Black-Scholes derivation assumes that the interest rate is always constant and the same for all maturites. Of course, the riskless interest rate is not constant, and bonds of different maturities have different yields. In the Black-Scholes formula, the interest rate always appears in $B = Xe^{-r_f T}$, which is the price of a discount bond with a face value of X maturing at the same time as the option. This discount bond price is what we should use in general. Traditionally, prices of Treasury securities were used, but now most practitioners use LIBOR instead because Treasuries are viewed as less liquid. In principle, the volatility should be adjusted as well for the volatility of the discount bond price and $\sigma^2 T$ should be replaced by the remaining volatility of the S/B,[5] However, this adjustment does not usually matter in practice except for long maturities in times of high interest volatility.

[5] More precisely, $\sigma^2 T$ should be replaced by $\mathrm{var}(\log(S_T/B_T) - \log(S_0/B_0))$.

What variance to use

The choice of variance estimate is even more important than the choice of interest rate in most option pricing applications. Getting the variance wrong can have a big impact on the computed price, and a big impact on the effectiveness of a hedge. The two most common methods of determining the appropriate variance are historical estimation and implied variance.

In the historical estimation approach, simply look at the variance of historical returns and then adjust for the length of the time horizon. For technical reasons, it is best to use log returns (that is, to compute the sample variance of $\log(1 + return)$). Be sure that your returns are computed correctly, and that you have adjusted properly for any dividends or stock splits. Weekly returns seem like an ideal choice: longer returns give less accurate estimates and are more subject to changes in variance over time, while shorter returns are more contaminated by the bid-ask spread, liquidity distortions, and non-trading ects. After the variance is computed, do the proper adjustment to convert to annual terms. For example, a weekly variance of returns of $.0032$ corresponds to an annual variance of $52 \times 0.0032 \approx 0.16$ or an annual standard deviation of $\sqrt{0.16} = 40\%$.

What variance to use (continued)

The implied variance approach is forward looking and is a way of reading market participants' assessment of variance. It requires there to be quoted prices available for options on the stock on which the option is written, or for options on a related instrument. The idea is to look at what variance (the implied variance) would be consistent with the option prices you see in the market. In doing so, it is important to get option and stock prices that are as nearly contemporaneous as possible, which is particularly difficult if the stock or its options are thinly traded. This approach is used mostly in active markets (for example, for stock index futures). Practitioners often use implied variances as an indicator of the state of these markets.

Of course, individual judgment can enter the equation. For example, if you believe that historical variances were estimated on a particularly turbulent period and that we are in a quieter period, then you should adjust the estimate downward. Or, if you obtain an implied variance that doesn't seem sensible, you should obviously consider whether the price quotes (and your calculations) are reliable or maybe you disagree with the other market participants.

Other models

Adjusting Black-Scholes for dividends can be done reasonably in several ways, for example but taking the present value of anticipated dividends out of the initial stock price (since you have an option on the value of what will be left over after dividends are paid between now and maturity). If volatility is a function of time, that can be handled by plugging in the total remaining volatility $\int_t = 0^T \sigma_t^s dt$ in place of $\sigma^2 T$. U sually it is more reasonable to think of the volatility changing randomly over time, and if this is important we require a different model, for example based on an ARCH model for volatility. Having possible jumps (discontinuities) in the stock price also requires a new model. With some strong assumptions, jumps or changing volatility can be handled as a mixture of Black-Scholes models.

"Greeks"

When this $\uparrow$	call price	put price
Stock price	$\uparrow$	$\downarrow$
Volatility	$\uparrow$	$\uparrow$
Strike price	$\downarrow$	$\uparrow$
Time to maturity	$\uparrow$	$\updownarrow$
Interest rate	$\uparrow$	$\downarrow$
Dividend	$\downarrow$	$\uparrow$

for any position Y:

delta $\delta = \partial Y / \partial S$ (like a beta β)
gamma $\gamma = \partial \delta / \partial S$
vega $\mathcal{V} = \partial Y / \partial \sigma^2$
rho $\rho = \partial Y / \partial r$
theta $\theta = \partial Y / \partial T$

Delta-hedging is a basic hedge. Hedging gamma and vega as well make it so you have to rebalance less often.

Warning: Black-Scholes is for underlying assets

Do not just plug in an interest rate, vol, or whatever where the stock price goes in Black-Scholes! If you make this common mistake, the result is nonsense and any decisions based on this mistake are likely to cost you money. The Black-Scholes price is based on going long or short the stock. However, there is no asset we can buy today whose price is the interest rate today and whose price tomorrow is the interest rate tomorrow!

We can go a long way by using Black-Scholes on an asset a final payoff equal to whatever we have an option on. For example, if we have an call option with strike price X on a stock index futures, we can think about how to replicate that futures payoff using an investment in stock today. In an ideal world without frictions, consider the price at time 0 of the call option maturing at time t on a futurers contract maturing at time T on a share of stock worth $\$50$ now. If the interest rate is constant equal to r and the stock pays no dividends, the futures price at time t will be $S_t e^{r(T-t)}$ and we can buy that claim by buying $e^{r(T-t)}$ shares of stock now worth $S_0 e^{r(T-t)}$. Therefore, the futures option is valued by Black-Scholes as $S_0 e^{r(T-t)} N(x_1) - X e^{-rt} N(x_2)$, where $x_1 = \log(S_0 e^{rT}/X)/(\sigma\sqrt{t}) + \sigma\sqrt{t}/2$ and $x_2 = \log(S_0 e^{rT}/X)/(\sigma\sqrt{t}) - \sigma\sqrt{t}/2$.

Using Black-Scholes to value a currency option

Suppose the exchange rate χ_t of Euros in terms of dollars has a constant variance σ^2 per unit time and we take the riskfree interest rate for dollars r and riskfree interest rate for Euros r^E both to be constant. (Taking the interest rates to be constant is usually a good approximation except for long maturity options in very volatile interest rate environment, since the interest rate volatility has a much smaller impact on pricing than the exchange rate volatility.) Then a European call option maturing at T on $1,000$ Euros has an identical payoff as a call option on a discount bond paying $1,000$ Euros at time T. The price of the discount bond today is $1000e^{-r^E T}$ and this is our "stock price" (price of the underlying asset) for applying Black-Scholes. Our bond price is for the domestic bond and equals Xe^{-rT}. So, we have that Black-Scholes can value the option as $1000e^{-r^E T}N(x_1) - Xe^{-rT}N(x_2)$ where $x_1 = \log(1000e^{(r-r^E)T}/X)/(\sigma\sqrt{T}) + \sigma\sqrt{T}/2$ and $x_2 = \log(1000e^{(r-r^E)T}/X)/(\sigma\sqrt{T}) - \sigma\sqrt{T}/2$.