

Fixed-Income Securities

Lecture 5: Tools from Option Pricing

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- Review of binomial option pricing
- Interest rates and option pricing
- Effective duration
- Simulation

Option-pricing theory

The option-pricing model of Black and Scholes revolutionized a literature previously characterized by clever but unreliable rules of thumb. The Black-Scholes model uses continuous-time stochastic process methods that interfere with understanding the simple intuition underlying these models. We will use instead the binomial option pricing model of Cox, Ross, and Rubinstein, which captures all of the economics of the continuous-time model but is simpler to understand and use. The original binomial model assumed a constant interest rate, but it does not change much when interest rates can vary.

Cox, John C., Stephen A. Ross, and Mark Rubinstein, 1979, Option Pricing: A Simplified Approach, *Journal of Financial Economics* 7, 229–263

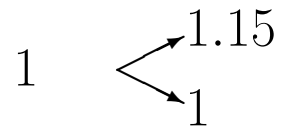
Black, Fischer, and Myron Scholes (1973), The Pricing of Options and Corporate Liabilities, *Journal of Political Economy* 81, 637–654

A simple option pricing problem in one period

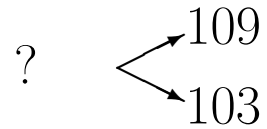
Short-maturity bond (interest rate is 5%):

$$1 \rightarrow 1.05$$

Long bond:



Derivative security (intermediate bond):



The replicating portfolio

To replicate the intermediate bond using α_S short-maturity bonds and α_L long bonds:

$$109 = 1.05\alpha_S + 1.15\alpha_L$$

$$103 = 1.05\alpha_S + 1.00\alpha_L$$

Therefore $\alpha_S = 60$, $\alpha_L = 40$, and the replicating portfolio is worth $60+40 = 100$. By absence of arbitrage, this must also be the price of the intermediate bond.

General single-period valuation

Compute the replicating portfolio and the price of the general derivative security below. Assume $U > R > D > 0$.

Riskless bond:

$$1 \rightarrow R$$

Long bond:

$$S \begin{cases} \rightarrow US \\ \rightarrow DS \end{cases}$$

Derivative security:

$$??? \begin{cases} \rightarrow V_U \\ \rightarrow V_D \end{cases}$$

State prices and risk-neutral (artificial) probabilities

Valuation can be viewed in terms of state prices p_U and p_D or risk-neutral probabilities π_U and π_D , which give the same answer (which is the only one consistent with the absence of arbitrage):

$$Value = p_U V_U + p_D V_D = R^{-1}(\pi_U^* V_U + \pi_D^* V_D)$$

where

$$p_U = R^{-1} \frac{R - D}{U - D} \quad p_D = R^{-1} \frac{U - R}{U - D}$$

are called *state prices* and

$$\pi_U = \frac{R - D}{U - D} \quad \pi_D = \frac{U - R}{U - D}$$

are called *risk-neutral probabilities*. Note that the risk-neutral probabilities equate the expected return on the two assets. Risk-neutral probabilities need not equal the true probabilities, but most term structure models used in practice assume they are the same.

Multi-period valuation

One special appeal of the binomial model is that multi-period valuation is not much more difficult than single-period valuation. This is because multiperiod valuation simply applies the single- period valuation again and again, stepping from maturity backwards through the interest-rate tree. Another approach that is not limited to the binomial model is to use a simulation for valuation. This is because we can express the value of a claim as an expectation using the artificial probabilities. For example, we have that

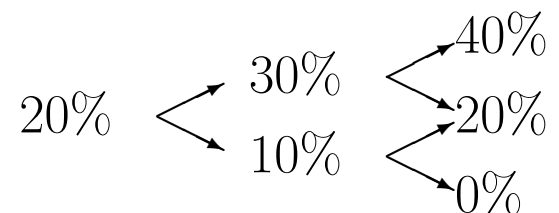
$$E^* \left[\frac{1}{R_1} \frac{1}{R_2} \frac{1}{R_3} \dots \frac{1}{R_T} C_T \right]$$

is the value at time 0 of a claim to the cash flow C_T to be received at time T , where R_s is one plus the spot rate of interest quoted at time $s - 1$ for investment to time s . To use simulation, we simulate many realizations of the interest rates and the final cash flow, and take the sample mean corresponding to population statistic in the formula.

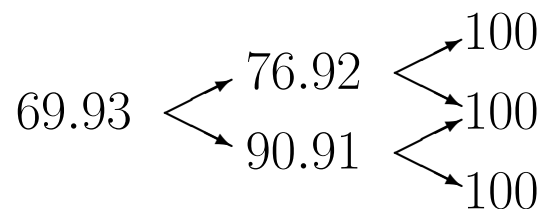
Example: valuation of a riskless bond

Consider a two-period binomial model in which the short riskless interest rate starts at 20% and moves up or down by 10% each period (i.e., up to 30% or down to 10% at the first change). The artificial probability of each of the two states at any node is $1/2$. What is the price at each node of a discount bond with face value of $\$100$ maturing two periods from the start?

The interest tree is



and the value of the bond is given by



Special considerations for pricing interest derivatives

- The interest rate is not constant and varies through the tree.
- The interest rate is not an asset price (and cannot, for example, be plugged in as the price in the Black-Scholes formula).
- The short rate is known at the *start* of the period and is forward-looking (while the return on a risky asset is not known until the end of the period).
- It is usually handiest to assume risk-neutral probabilities and the volatility and work from there.
- We usually match the initial observed yield curve one way or the other.
- One implicit or explicit input is a view on mean reversion and volatility of interest rates.

Valuation of interest derivatives

We value interest derivatives besides the riskless bonds using the same approach. First we compute the value at maturity based on the contractual terms, and then we compute the value at previous points in the tree using repeatedly the formula for one-period valuation. If we have an early exercise option or a cash flow before maturity, that needs to be taken into account in the one-period valuations. We need to be careful to understand what is shown in the tree, for example, perhaps it is the value of a live bond (that is not converted) after the coupon interest in the period is received.

Sometimes we need to value one security first before valuing another. For example, if we are computing the value of an option on a coupon bond, we need to value the bond at each node in the tree before valuing an option on the bond. To include the possibility of exercise, the value of the option will be the larger of the value if not exercised (which comes from looking at next period's value in the option pricing formula) and the value if exercised now (which comes from looking at the bond price at this node). Care must be taken to decide whether we need to look at the bond price before or after any coupon is paid.

In-class exercise: bond and bond option valuation

Consider a two-year binomial model. The short riskless interest rate starts at 50% and moves up or down by 25% each year (i.e., up to 75% or down to 25% at the first change). The artificial probability of each of the two states at any node is $1/2$. What is the price at each node, of a discount bond with face value of \$100 maturing two periods from the start?

What is the value at each node of an American call option on the discount bond (with face of \$100 maturing two periods from now) with a strike price of \$60 and maturity one year from now?

Fitting an initial yield curve: using fudge factors

Suppose we write down our favorite term structure model and we find that it does not even give the correct prices for Treasury strips. This is more than a minor embarrassment, since it means that the pricing of derivative securities from the model will produce an arbitrage in reality. Fortunately, we can use *fudge factors* to correct the pricing of the discount bonds. Fudge factors are adjustments to the interest rate process. The same adjustment is made to all nodes at the same point in time.

This approach was introduced in my paper,¹ which is why they are sometimes called “Dybvig factors” (for example in the BARRA documentation).

¹P. Dybvig, Bond and Bond Option Pricing Based on the Current Term Structure, 1997, Mathematics of Derivative Securities, Michael A. H. Dempster and Stanley Pliska, eds., Cambridge University Press.

Fudge factors in the binomial model

In the binomial model, we can write the discount bond prices as

$$D^{om}(0, t) = E^* \left[\frac{1}{R_1^{om}} \frac{1}{R_2^{om}} \frac{1}{R_3^{om}} \cdots \frac{1}{R_t^{om}} \right]$$

where om indicates the original model and R_s^{om} is one plus the interest rate at time s in that original model. If we see instead discount rates $D(0, t)$ in the economy, we want to change the interest rate process to fit what is observed. Intuitively, we want to add to each interest rate the different between the implied forward rate in the economy and the interest rate in the original model (and this adjustment is exact enough for many purposes). More precisely, we set

$$R_s = R_s^{om} \frac{D(0, s-1)/D(0, s)}{D^{om}(0, s-1)/D^{om}(0, s)}$$

Note that the numerator in the adjustment is one plus the forward rate observed in the economy, and the denominator is one plus the forward rate in our original model.

In-class exercise: fudge factors

Consider a two-year binomial model. Start with an original model in which the short riskless interest rate starts at 5% and moves up or down by 5% each period (i.e., up to 10% or down to 0% at the first change). The artificial probability of each of the two states at any node is $1/2$.

What is the price of a one-year discount bond in this original model? the two-year discount bond?

Suppose the one-year discount rate in the economy is 6% and the two two-year discount rate is 7%. Compute the fudge factors and draw the tree for the adjusted interest rate process.

Fudge factors: good and bad points

The fudge factors do fit today's STRIP curve of riskless bonds, but may fail to do so tomorrow. When using fudge factors, it is necessary to re-estimate the fudge factors every period. To the extent that the fudge factors are changing a lot over time, there is significant volatility in interest rates that is not part of the option pricing activity. Understating the volatility tends to underprice options and long bonds with a lot of convexity.

Fortunately, most of the movements in the yield curve are explained well by a single risky factor, so viewing the rest as a total surprise (by including it as updates in the fudge factors) will not usually create big pricing errors. This is one result that surprised me in my analysis: I originally planned to criticize models that implicitly used fudge factors, but I actually found some support for what they were doing. It pays to keep an open mind! I should note that it is possible to design securities (for example an option on a spread between yields) that eliminate the main risky factor and will be badly mispriced using fudge factors. This idea may come in handy at some point when you are trading with people using fudge factors in a mechanical way!

The main alternative to using fudge factors is the use of multi-factor models that include different sources of interest risk. We might have one factor for more-or-less parallel shifts, another for changes in the slope, and a third for curvature. These models can be handled well by simulation (and in some cases we have exact formulas), but tend to get messy in a binomial framework.

Mean reversion in interest rates

The simple binomial model with equal up and down probabilities at each node has several unrealistic features. One is that futures prices in the model have the same sensitivity to rate shocks as the short rate, while actual short rates move much more than forward rates. Or, to put it another way, short rates are much more sensitive to interest shocks than are yields on long bonds.

We remedy this problem by introducing mean reversion. Interest rates move on average towards a long-term mean. If interest rates move up above the mean, then the mean change becomes negative and rates tend to move back (revert) towards the mean. The usual form for mean reversion is

$$E[r_{t+1} - r_t] = k(\bar{r} - r_t)$$

When the interest rate r_t is larger than the long-term mean \bar{r} , then the interest rate usually declines. When r_t is smaller than \bar{r} , then the interest rate usually rises.

Mean reversion in the binomial model

In the binomial model, suppose that the interest rate increases from r_t to $r_t + \delta$ with probability π and decreases from r_t to $r_t - \delta$ with probability $1 - \pi$. Then

$$\begin{aligned} E[r_{t+1} - r_t] &= \pi\delta + (1 - \pi)(-\delta) \\ &= (2\pi - 1)\delta \end{aligned}$$

For mean reversion (using the formula on the previous slide), we want to set this mean return equal to $k(\bar{r} - r_t)$. Solving

$$(2\pi - 1)\delta = k(\bar{r} - r_t)$$

for π , we obtain

$$\pi = \frac{1}{2} + \frac{k(\bar{r} - r_t)}{2\delta}$$

Without mean reversion, $k = 0$ and $\pi = 1/2$. With mean reversion, $k > 0$ and the probability of going up is less than $1/2$ when r is large but smaller than $1/2$ when r is small.

Mean reversion: a technical adjustment

If the probability on the last slide (which we use for the risk-neutral probability) is computed to be larger than 1, we round it down to 1. And, if it is computed to be smaller than 0, we round it up to 0. Having artificial probabilities larger than 1 or less than 0 does not make economic sense (it implicitly implies arbitrage), and may lead to numerical instability. As a computational boon, we do not have to compute security prices for nodes beyond where the probability goes to 1 or 0, since nodes further out do not carry any weight in valuation.

Volatility

As in equity options, changing volatility is an important concern for pricing of interest derivatives. The volatility of interest rates tends to be higher when interest rates are higher, but there is not a tight connection between volatility of rates and their levels. As a first adjustment for changing volatility, we can make the interest rate grid more widely spaced at large rates and more narrowly spaced at small rates. This has the advantage of keeping the interest rate positive and also admitting that volatility tends to be higher when rates are higher.

It is possible to use a binomial model with changes in volatility as well as rates, but it is usually simpler to use a simulation model in this case. Adding interest rate volatility has a minor impact on the programming difficulty in simulation models but a major impact in binomial models. The only drawback is that simulation models tend to be much slower. I like to use between *100,000* and *1,000,000* sample paths for simulation estimates of derivative prices; practitioners often save computer time by using fewer paths and obtain inaccurate results. Computer time is getting cheaper and cheaper, and there is less and less excuse for making this mistake.

Volatility models

There are two main models of uncertain volatility used in econometrics. Both have names you can love to hate. *Stochastic volatility* (SV) models have a volatility at a point in time that is unobserved but can only be guessed from the history. The name is inappropriate because it sounds like it could apply to any model of volatility. *ARCH* models look at the best estimate of volatility conditional on the history. ARCH models are easy to work with and in spite of appearances are no less general than SV models (since volatility in the ARCH model at a point in time can be interpreted as the best estimate of the unknown volatility in an SV model). ARCH stands for AutoRegressive Conditional Heteroskedasticity; do I have to explain why I do not like this term? There is a whole alphabet soup of various ARCH models, for example GARCH (Generalized ARCH, which is actually less general than ARCH), LARCH (Linear ARCH), EGARCH (Exponential GARCH), etc.

Sample SV and ARCH models

Here is an example of an SV model:

$$\begin{aligned}r_{t+1} &= r_t + k_r(\bar{r} - r_t) + \sigma_t e_{r,t} \\ \sigma_{t+1} &= (\sigma_t + k_\sigma(\bar{\sigma} - \sigma_t))(1 + e_{\sigma,t})\end{aligned}$$

Note that there is a separate error term in the volatility equation, and we do not know the true volatility.

Here is an example of an ARCH model:

$$\begin{aligned}r_{t+1} &= r_t + k_r(\bar{r} - r_t) + \sigma_t e_{r,t} \\ \sigma_{t+1} &= (\sigma_t + k_\sigma(\bar{\sigma} - \sigma_t)) |e_{r,t}|\end{aligned}$$

In this case the volatility equation does not have its own error term and the random part of the change in volatility depends on the error term (the shock) in the interest rate term. If we know the parameters and the starting volatility, we can figure out the volatility for all times from observing the interest rates. Knowing the volatility in the model simplifies econometric estimation of the parameters and also makes application easier.