Fixed-Income Securities
Lecture 6: Symphony of In-class Exercises
Xtreme Review

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- Concepts
- Definitions
- Formulas
- Practice practice practice

Some Central Ideas

- arbitrage: exact and approximate
- pricing using replication
- hedging using replication
- formulas and intuitions relating rates
- traditional approaches
  - matching cash flows
  - duration
  - effective duration (mean reversion, slope and twist of yield curve)
- modern option pricing tools
  - binomial model (fast, single source of noise)
  - simulation (slow, many sources of noise, not good for American)

Basic Notation

spot rate r_t**:** quoted at $t-1$ for borrowing/lending from $t-1$ to t

forward rate $f(s,t)$**:** quoted at s for borrowing/lending from $t-1$ to t

discount factor $D(s,t)$**:** price at s of receiving 1 at a future date t

zero-coupon rate $z(s,t)$**:** yield at s for a zero-coupon bond maturing at t

par coupon rate $c(s,t)$**:** coupon rate ($=$ yield) quoted at s for a coupon bond maturing at t and trading at par

present value PV**:** value today of a series of future cash flows

present value NPV**:** value today of a series of future cash flows, less the initial price

Basic Formulas

$$r_t = f(t-1, t)$$

$$D(s,t) = \frac{1}{\Pi_{s=1}^{t}(1 + f(0,s))}$$

$$f(s,t) = \frac{D(s,t-1)}{D(s,t)} - 1$$

$$D(s,t) = \frac{1}{(1 + z(s,t))^{t-s}}$$

$$z(s,t) = D(s,t)^{-(t-s)} - 1$$

$$c(s,T) = \frac{1 - D(s,T)}{\Sigma_{t=s+1}^{T} D(s,t)}$$

$$PV = \sum_{s=1}^{t} D(0,s)c_s$$

$$\begin{aligned} NPV &= PV - P \\ &= \sum_{s=0}^{t} D(0,s)c_s \end{aligned}$$

In-class Exercise: computations using zero-coupon bond prices

Suppose a zero-coupon bond maturing one period out (at time 1) has a price of $\$90$ and a zero-coupon bond maturing two periods out (at time 2) has a price of $\$80$, both per $\$100$ of face value.

1. Compute the discount factors $D(0,1)$ and $D(0,2)$.
2. Compute the forward rates $f(0,1)$ and $f(0,2)$.
3. If we can borrow forward at 10% from one year out until two periods out, what is the arbitrage?

In-class Exercise: computations using zero-coupon rates

Suppose the yield on a one-year discount bond is 10% and the yield on a two-year zero-coupon bond is 12%.

1. Compute the discount factors $D(0,1)$ and $D(0,2)$.
2. Compute the forward rates $f(0,1)$ and $f(0,2)$.
3. What is the price of a two-year coupon bond with a face value of $\$500$ and a coupon rate of 20%?

In-class Exercise: matching cash flows

Suppose the price is $\$212$ for a 2-year coupon bond with face of $\$200$ and an annual coupon (first one one year from now) of $\$40$. Suppose also that the price is $\$150$ for a 1-year coupon bond with face of $\$150$ and an annual coupon (one remaining one year from now) of $\$15$.

Remaining pension benefits in a plan having two more years to go are $\$95,000$ one year from now and $\$60,000$ two years from now. What replicating portfolio of the two coupon bonds covers the pension liabilities exactly? What is the price of the replicating portfolio?

Formulas connecting rates

$$z(0,T) = \left(\prod_{s=1}^{T}(1+f(0,s))\right)^{1/T} - 1 \approx \frac{1}{T}\sum_{s=1}^{T} f(0,s)$$

The zero-coupon rate is an average of forward rates up to that maturity.

$$c(0,T) = \sum_{s=1}^{T} w(0,s) f(0,s)$$

where

$$w(0,s) = \frac{D(0,s)}{\sum_{t=1}^{T} D(0,t)}$$

The par-coupon rate is a weighted average of forward rates up to that date, with more weight on the earlier maturities.

In-class Exercise: formulas connecting rates

Suppose the spot rate is 5% and the forward rate one year out is 6%. What are the one- and two-year zero-coupon rates? What are the one- and two-year par-coupon rates?

Duration formulas

traditional (Macauley) duration:

$$duration = \sum_{t-1}^{T} \frac{c_t D(0,t)}{\Sigma_{s=1}^{T} c_s D(0,s)} \ t$$

The discount factors $D(0,t)$ are usually computed using either the bond's yield (i.e. $D(0,t) \equiv 1/(1+y)^t$) or using the actual discount factors. Macauley duration assumes that random shocks impact forward rates equally at all maturities.

effective duration (sens = short for sensitivity):

$$sens(\textit{effective duration}) = \left(\frac{\Sigma_{s=1}^{T} sens(s) c_s D(0,s)}{\Sigma_{s=1}^{T} c_s D(0,s)} \right)$$

For effective duration, shocks affect different forward rates differently, so the amount of interest rate exposure is no longer proportional to time-to-maturity, even for a discount bond.

the particular effective duration measure we have used:

$$sens(duration) = \exp(-.125 * duration)$$

$$duration(sens) = -\log(1 - .125 * sens)/.125$$

In-class exercise: duration and effective duration

Suppose the yield curve today is flat at 5%. Compute the duration and effective duration of a portfolio paying $\$100$, 10 years from now, and $\$100$, 20 years from now. Recompute the duration and effective duration assuming a flat yield curve at 10%.

Option pricing formulas

single-period:

$$\text{Value} = R^{-1}(\pi_U V_U + \pi_D V_D)$$

expected present value computed using artificial "risk-neutral" probabilities... Risk-neutral probabilities could be computed from the payoffs of some asset, but more commonly we make assumptions about them directly.

multi-period:

$$\text{Value} = E^* \left[\frac{1}{R_1} \frac{1}{R_2} \frac{1}{R_3} \ldots \frac{1}{R_T} C_T \right]$$

This formula is especially useful for simulations but can also be used in simple binomial cases without American or conversion features.

In-class exercise: binomial model

The spot interest rate is 5%. Each year it goes up by 5% (e.g. from 5% to 10%) with risk-neutral probability $1/3$ or down by 2% (e.g. from 5% to 3%) with risk-neutral probability $2/3$. What is the price of a 2-year interest-rate cap with a capped rate of 5%?

Mean reversion and fudge factors

For mean reversion

$$E[r_{t+1} - r_t] = k(\overline{r} - r_t)$$

in the binomial model with equal changes δ or $-\delta$ in rates, set

$$\pi_U = \frac{1}{2} + \frac{k(\overline{r} - r_t)}{2\delta}$$

Without mean reversion, $k = 0$ and $\pi_U = 1/2$.

fudge factors: To match actual discount factors $D(0,t)$, modify the original model—om—as follows:

$$R_s = R_s^{om} \frac{D(0,s-1)/D(0,s)}{D^{om}(0,s-1)/D^{om}(0,s)}$$

or approximately

$$r_s = r_s^{om} + f(0,s) - f^{om}(0,s)$$

In-class exercise: capstone problem with fudge factors and mean reversion

Consider a two-year binomial model. Start with an original model in which the short riskless interest rate starts at 5% and moves up or down by 2.5% each period (i.e., up to 7.5% or down to 2.5% at the first change). The artificial probability of each of the two states at any node is determined by whatever makes mean reversion k equal to 20% per year with a long-term mean of 5%.

What is the price of a one-year discount bond in this original model? the two-year discount bond?

Suppose the actual one-year discount rate in the economy is 6% and the actual two-year discount rate is 6.5%. Compute the fudge factors and draw the tree for the adjusted interest rate process.