FIN 550 Exam, June 25, 2008

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This is a closed-book examination. Answer all questions as directed. Mark your answers directly on the examination. Make sure each answer is clearly indicated. There are no trick questions on the exam. Good luck!

1. True-False (24 points)

A. Linear programs typically have interior solutions.

False. Unless the objective is zero, all solutions are at the boundary.

B. A local optimum of a convex optimization problem is a global optimum.

True.

C. The Fundamental Theorem of Asset Pricing is about the absence of arbitrage.

True.

D. A grid search is faster but less reliable than the Newton method.

False. The Newton method is faster but less reliable than a grid search.

E. The value function in a dynamic program is the value of continuing using the optimal strategy.

True.

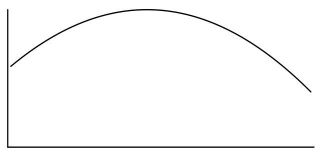
F. Slack variables are used to convert inequality constraints into equality constraints.

True.

2. Multiple choice – Univariate optimization (16 points)

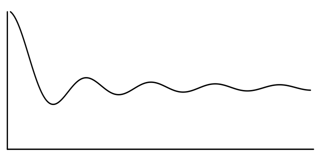
Choose the answer that best describes each picture

A.



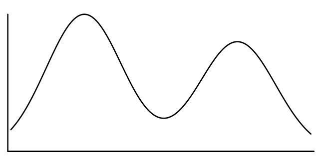
(i) strictly convex function
(ii) strictly concave function
(iii) linear function
(iv) none of the above

B.



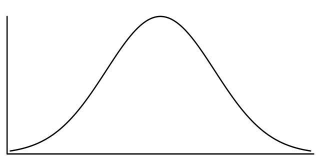
(i) strictly convex function
(ii) strictly concave function
(iii) linear function
(iv) none of the above

C.



(i) no local maximum
(ii) one local maximum
(iii) two local maxima
(iv) three local maxima

D.



(i) no local maximum
(ii) one local maximum
(iii) two local maxima
(iv) three local maxima

3. Linear Programming (30 points)

Consider the following linear program:

Choose nonnegative x_1, x_2, and x_3 to
maximize $2x_1 + x_2 + 3x_3$, subject to
$x_1 + 2x_2 + 3x_3 \leq 6$ and
$x_1 + x_2 \leq 3$

A. What is the dual linear program?

Choose nonnegative y_1 and y_2 to
minimize $6y_1 + 3y_2$, subject to
$y_1 + y_2 \geq 2$,
$2y_1 + y_2 \geq 1$, and
$3y_1 \geq 3$.

B. Is the primal feasible? Is the dual feasible?

Yes to both. $(0, 0, 0)$ is feasible in the primal and $(1, 1)$ in the dual.

C. Infer from the answers in part B: Is the primal bounded? Is the dual bounded?

Yes, because primal feasible iff dual bounded and dual feasible iff primal bounded.

D. Solve the dual problem.

The second constraint is redundant. We know from parts B and C there is a solution and from first principles that the solution has to be at an extreme point of the feasible set. Graphing the constraints shows that extreme points in the feasible set are $(1, 1)$ and $(2, 0)$; $(1, 1)$ is the solution because it has the smaller value $9 < 12$.

E. Use the solution to the dual problem to solve the primal problem.

Since the middle constraint in the dual is not binding at the solution, its shadow price is zero so that $x_2 = 0$ in the solution to the primal. Also, y_1 and y_3 are not zero so the shadow prices of the first and third constraints in the primal are positive and these constraints must be binding. Solving $x_1 + 3x_3 = 6$ and $x_1 + 0 = 3$, we have $x_1 = 3$ and $x_3 = 1$. Therefore the solution is $(x_1, x_2, x_3) = (3, 0, 1)$. This has value 9, the same as the value of the dual. Since we have found feasible solutions to the primal and the dual with the same value, they must be solutions to the problems, giving an independent check on the analysis.

4. Kuhn-Tucker Conditions (30 points)

Consider the following optimization problem:

Choose c_u and c_d to
maximize $\frac{1}{2}\log(c_u) + \frac{1}{2}\log(c_d)$, subject to
$\frac{4}{5}(\frac{1}{4}c_u + \frac{3}{4}c_d) \leq 6$.

This is a single-period choice of investment for consumption in a binomial model with log utility, initial wealth of 6, actual probabilities 1/2 and 1/2, risk-neutral probabilities 1/4 and 3/4, and riskfree rate of 25% (and therefore discount factor 4/5).

A. What are the objective function, choice variables, and constraint?

objective function: $\frac{1}{2}\log(c_u) + \frac{1}{2}\log(c_d)$
choice variables: c_u and c_d
constraint: $\frac{4}{5}(\frac{1}{4}c_u + \frac{3}{4}c_d) \leq 6$

B. What are the Kuhn-Tucker conditions?

$(\frac{1}{2c_u}, \frac{1}{2c_d}) = \lambda(\frac{1}{5}, \frac{3}{5})$, $\lambda \geq 0$, and
$\lambda(\frac{4}{5}(\frac{1}{4}c_u + \frac{3}{4}c_d) - 6) = 0$

C. If we add constraints $c_u \geq 6$ and $c_d \geq 6$, what are the Kuhn-Tucker conditions now?

$(\frac{1}{2c_u}, \frac{1}{2c_d}) = \lambda(\frac{1}{5}, \frac{3}{5}) + \lambda_u(-1, 0) + \lambda_d(0, -1)$,

λ, λ_u, and $\lambda_d \geq 0$, $\lambda(\frac{4}{5}(\frac{1}{4}c_u + \frac{3}{4}c_d) - 6) = 0$,
$\lambda_u(6 - c_u) = 0$, and
$\lambda_d(6 - c_d) = 0$.

5. Bonus question (30 bonus points)

A. Solve the optimization problem in problem 4 without the extra constraints in part 4C.

There is more than one way of solving this problem and the next one.

$\frac{1}{2c_u} = \lambda\frac{1}{5} \Rightarrow c_u = \frac{5}{2\lambda}$
$\frac{1}{2c_d} = \lambda\frac{3}{5} \Rightarrow c_d = \frac{5}{6\lambda}$
substituting into the budget constraint, which must be binding or else we could increase the objective function by increasing consumption:: $\frac{1}{5}\frac{5}{2\lambda} + \frac{3}{5}\frac{5}{6\lambda} = 6 \Rightarrow \lambda = \frac{1}{6}$. therefore: $c_u = 15$ and $c_d = 5$ which can easily be verified to satisfy the K-T conditions for $\lambda = 1/6$. Since this is a convex optimization with a strictly concave objective function and the KT conditions are not degenerate (the gradients of binding g's are linearly independent), this is the solution.

B. Solve the optimization problem in problem 4 with the extra constraints in part 4C.

We know the budget constraint is binding since otherwise increasing consumption will increase value. Since the constraint $c_d \geq 6$ is violated in the solution in part A and $c_u \geq 6$ is satisfied and not binding, it is reasonable to conjecture that $c_d \geq 6$ is the only new constraint binding in the solution here.[1] Then the budget constraint implies $c_u = 12$. To prove this is a solution, we solve for the multipliers. Since the constraint $c_u \geq 6$ is not binding, the c. slackness condition $\lambda_u(6 - c_u) = 0$ implies $\lambda_u = 0$ and therefore
$\frac{1}{2c_u} = \lambda\frac{1}{5} \Rightarrow \lambda = \frac{5}{24}$.

[1] Of course, this can be derived by going through cases of what constraints are binding. If we assume both new constraints are binding, $c_u = c_d = 6$, which does not satisfy the budget constraint. If we assume neither new constraint is binding, we obtain the solution in part A, which is not feasible. If we assume the new constraint $c_u \geq 6$ is binding but the other new constraint is not, we find that $\lambda_u < 0$, which is not consistent with the first-order conditions.

Then we can also solve for λ_d:
$\frac{1}{2c_d} = \lambda\frac{3}{5} - \lambda_d \Rightarrow \lambda_d = \lambda\frac{3}{5} - \frac{1}{2c_d} = \frac{5}{24}\frac{3}{5} - \frac{1}{12} = \frac{1}{24}$.
It is easy to verify that the KT conditions are satisfied by choosing $c_u = 12$, $c_d = 6$, $\lambda = 5/24$, $\lambda_u = 0$, and $\lambda_d = 1/24$. Since this is a nondegenerate solution of the KT conditions for a convex optimization with a strictly concave maximand, $(c_u, c_d) = (12, 6)$ is the unique solution.