Problem Set 1: Kuhn-Tucker Conditions and Binomial Portfolio Optimization
Financial Optimization
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1. Optimal portfolio choice in the binomial model

Recall that in the simplest standard binomial model, in each period the riskless asset pays off $R > 1$ per dollar invested and the stock, the risky asset, pays off $U > R$ in the good state and $D < R$ in the bad state. Valuation can be done using the risk-neutral probabilities $\pi_U^* = (R - D)/(U - D)$ and $\pi_D^* = 1 - \pi_U^* = (U - R)/(U - D)$ while preferences are based on actual probabilities π_U and $\pi_D = 1 - \pi_U$. We take U, D, R, and π_U to be the same at all nodes, and we assume there is a positive risk premium so that $\pi_U > \pi_U^*$.

A state of nature is characterized by the sequence of up and down moves. All of the variables we are interested in will be path independent, so we will think of a collapsed tree and consider a final state characterized by the number of ups n and number of downs $N - n$, where N is the total number of periods. Then the final stock price is

$$S_N(n) = S_0 U^n D^{N-n}.$$

Having n up moves over N periods has risk-neutral probability

$$\pi^*(n, N) = \left(\begin{array}{c} n \\ N \end{array} \right) {\pi_U^*}^n {\pi_D^*}^{N-n},$$

and actual probability

$$\pi(n, N) = \left(\begin{array}{c} n \\ N \end{array} \right) {\pi_U}^n {\pi_D}^{N-n},$$

where

$$\left(\begin{array}{c} n \\ N \end{array} \right) \equiv \frac{N!}{n!(N-n)!}$$

is the *binomial coefficient* giving the number of paths with n up moves over N periods.

Assume a von Neumann-Morgenstern utility function of the form

$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$

for $c > 0$ where $\gamma > 0$ and $\gamma \neq 1$. These are called CRRA (constant relative risk aversion[1]) or isoelastic preferences. For $\gamma = 1$, we would substitute log utility. For all CRRA preferences, $u'(c) = c^{-\gamma}$.

A. Set up an optimization problem to maximize expected utility of terminal consumption subject to the budget constraint that the expected present value of terminal consumption under the risk-neutral probabilities no greater than the initial wealth W_0.

B. Write down the Kuhn-Tucker conditions for the optimization problem.

C. Show that the optimal solution has the form $c_N(n) = K_N(S_N(n))^\alpha$. Compute α.

D. (extra for experts) Compute K_N. Show that the optimal strategy places the same proportion in the risky asset at all points in time.

2. Portfolio Insurance

A. Write down the same problem as in part 1 above but with the additional constraint that terminal consumption is no smaller than initial wealth.

B. Write down the Kuhn-Tucker conditions for this problem.

C. Show that the optimal solution is of the form $c_N(n) = \max(W_0, \hat{K}_N(S_N(n))^\alpha)$, where α is the same as in part 1.

D. (extra for experts) Write down a nonlinear equation that $\hat{K}_N$ must sat-

[1] This comes from the Arrow-Pratt measure of risk aversion. These utility functions have the same preference for relative (proportional) gambles independent of the wealth level.

isfy. Assuming reasonable values for U, D, R, π_U, N, and γ, compute $\hat{K}_N$ numerically.

3. (extra for experts) Write down the problem and Kuhn-Tucker conditions corresponding to (1) for the continuous-time model with constant riskfree rate r and a stock with constant mean $\mu > r$ and variance σ^s of return per unit time.

4. (challenger) Solve the problem in part 1 or 3 above using the utility function

$u(c) = \max(1, 2\sqrt{c})$.

Warning: this is a nonconvex problem and you cannot just solve the first-order conditions.