

Problem Set 4: Univariate Optimization and Solving for a Root
P. Dybvig

At the start of class next week, submit only problem 4 for grading.

Problems 1 and 2 are relatively simple problems about the algorithms, and something like these (with less calculation) might be on the final. Problems 3 and 4 are more involved problems that are similar to each other.

1. Convex and nonconvex optimization: grid search versus binary search. Recall that an optimization problem is called convex if

- The feasible set is convex, and
- The objective function is concave if maximizing or convex if minimizing.

For more information about convex sets and functions, see Appendix A in the book.

A. Which of the following univariate optimization problems are convex optimizations and which are not? Justify your answers.

(i)
Choose $x \in \Re$ to
maximize $x - x^2$, subject to
 $x \geq 1$ and
 $x \leq 6$.

(ii)
Choose $y \in \Re$ to
maximize $y - \exp(y)$, subject to
 $y^2 \geq 1/2$ and
 $y^2 \leq 25$.

(iii)
Choose $z \in \Re$ to
minimize $z \exp^{-z}$, subject to

$z \geq 1/2$ and
 $z \leq 10$.

B. Suppose it is important to find a global optimum. For which problems would you recommend using a grid search and for which would you recommend a binary search? (Assume you want to use the faster procedure if both are reliable.)

C. Follow the advice in B and solve each of the optimization problems.

2. Newton's method.

Consider finding a zero of the function

$$f(x) = \frac{x}{1+x^2}.$$

We know by inspection that $x = 0$ is the only zero of the function; let us see how Newton's method performs.

A. Compute $f'(x)$. Compute the rule for a Newton step: $\mathcal{N}(x) = x - f(x)/f'(x)$.

B. For what range of x does a sequence of normal Newton steps converge to the zero of f ? (You can do this numerically or using algebra.)

3. Portfolio Insurance. You are working for a big international investment bank, and you have a client that is interested in using portfolio insurance or some variant for a pension fund with \$1 billion in assets. (This is large compared with the entire size of some smaller firms, but only a small piece of the pension money at some of the largest firms.) Traditional portfolio insurance follows a dynamic trading strategy (computed using option pricing techniques) to generate a payoff at the end that is the larger of a guarantee G (often chosen as $G = W_0$) and some constant k times the value M at the end of investing \$1 in a stock index:

$$\text{payoff} = \begin{cases} G & kM \leq G \\ kM & G < kM \end{cases}$$

Sometimes a program is given a pool of money to invest in the stock (so $k = W_0$) and the put represented by the insurance (the option to forfeit the stock portfolio at the end in exchange for W_0) is paid for separately. More often, the price of the insurance is built into the cost of the program and k is chosen less than W_0 so the portfolio insurance payoff is worth W_0 . That is, k solves the nonlinear equation that says k plus the value of the put equals W_0 .

Recall that the the Black-Scholes call option pricing formula says the call price is

$$C(S, B, v) = SN\left(\frac{\log(S/B)}{v} + \frac{v}{2}\right) - BN\left(\frac{\log(S/B)}{v} - \frac{v}{2}\right),$$

where S is the stock price, B is the price of the discount bond paying X at the maturity of the option, and v is the square root of the remaining variance in $\log(S/B)$ between now and the maturity of the option. In the original Black-Scholes model, the relative stock price variance was a constant σ^2 per unit time, the interest rate was a constant r , and the strike price was X . Therefore, if the time-to-maturity is T , we can write $B = Xe^{-rT}$ and $v = \sqrt{\sigma^2 T}$. Recall that $\partial C(S, B, v)/\partial S = N(\frac{\log(S/B)}{v} + \frac{v}{2})$: all the terms from the chain rule having $\partial \log(S/B)/\partial S$ inside $N(\cdot)$ cancel (although this is a challenge to prove).

You should use the Black-Scholes model to solve for k in the portfolio insurance contract, assuming that the price of insurance is built-in.

A. Use put-call parity to write down the Black-Scholes put price + stock price.

B. Write down the nonlinear equation to be solved by k . Note that the stock price to be used in the put formula will depend on k .

C. Solve for the appropriate value for k . Assume $W_0 = \$1,000,000,000$, $\sigma = 0.2$, $T = 5$ years, and $r = 5\%/year$. Solve for k five times, for G taking on five different values ranging from $0.5W_0$ to $1.5W_0$. Diagnose and to the extent possible correct for any failures in the numerical solution. Use Solver

if you like or use Newton's method and do it yourself.

4. Generalized Portfolio Insurance. Traditional portfolio insurance is not so popular anymore, but there are some programs offering more general payoffs as a function of the underlying portfolio. One popular type of payoff is some sort of S-shaped payoff that gives up return on the upside in exchange for the insurance on the downside. We will assume a particular form of this with $k = 1$:

$$\text{payoff} = \begin{cases} G & W_0M \leq G \\ W_0M & G < W_0M \leq L \\ L & L < W_0M \end{cases}$$

where M is the payoff per dollar invested in the market, and L is the limit (cap) on the payoff.

A. Using the Black-Scholes model, write down the price of the insurance and the cap.

B. Suppose we are given the guarantee G . Write down a nonlinear equation to solve for the limit L that makes the payoff a fair investment that can be financed by a dynamic trading strategy.

C. Using the parameter values for problem 3C above, solve for the appropriate value for L . Diagnose and to the extent possible correct for any failures in the numerical procedure.

D. Extra for experts. Assume a simple mean-reverting Vasicek model for interest rates, so that

$$dr_t = \kappa(\bar{r} - r_t)dt + \sigma_r dZ_t$$

in the risk-neutral probabilities where $\kappa = .125$, $\bar{r} = .05$, and $\sigma_r = .01/\text{year}/\text{year}$ (in a typical year the spot interest rate might move from 4%/year to 3%/year or 5%/year). Assume that the instantaneous relative stock price variance is

constant with $\sigma = 0.2$ and the local correlation between stock price and interest movements is -5% . Then redo the calculations in part C under these assumptions.

E. Challenger. Give assumptions on preferences and the mean stock return and interest rate changes in the actual probabilities that would make the payoff in this problem the optimal choice.