

Problem Set 5: Dynamic Programming  
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At the start of class next week, submit only problem 4 for grading.

Problems 1 and 2 are relatively simple problems about dynamic programming, and something like these (with less calculation) might be on the final. Problems 3 and 4 are more involved problems that are similar to each other.

1. (short answer, no formulas) Suppose you want to trade on the discovery that the average movement in corn futures can be predicted by the last month's change in corn and soybean futures prices. Think about setting up a dynamic program to derive an optimal trading rule.

A. What choice variable(s) will your problem have?

B. What will be the objective function?

C. What will be the constraints?

D. What are the state variables for your problem?

2. (true-false) Dynamic programming concepts

A. The value function gives the value (under the optimal strategy) of the continuation of the problem.

B. The maximum principle says the solution of the dynamic program also solves a one-period problem that maximizes the value that comes directly from today's price plus the indirect effect in the future as measured by the value function.

C. Adding state variables is useful because state variables convert inequality constraints into equality constraints.

D. Time should not be included as a state variable because it is not random.

3. When to terminate a business. Our large company owns a restaurant. Its

annual cash inflows are random and can take on three values: \$1.8 million, \$1.2 million, and \$0.6 million, and the cash outflow is fixed and equal to \$1.3 million. Each year the cash inflow, whatever it is, stays the same with probability 0.8, or changes with probability 0.2. If the cash inflow this year is high or low (\$1.8 million or \$0.6 million), any move will be to the middle value (\$1.2 million). If the cash flow is at the middle value, it may move to the high value or the low value with probability 0.1 each.

At the beginning of each period, we have the option of closing the restaurant (permanently), in which case the value is zero because cash flows from that period onward are all zero. We want to maximize expected present value of remaining cash flows for all future dates, using a discount rate of 4%/year.

A. Write down the choice problem for when to close the restaurant.

B. What are the state variables for this problem?

C. Write down the Bellman equation for the problem.

D. Solve the Bellman equation.

4. Drilling for oil! Oil prices can be \$50/barrel, \$150/barrel, or \$200/barrel. Drilling until you strike oil is uncertain, but we will only concern ourselves with the average cost, and in the area where you are planning to drill on average it costs \$1 million to make a strike that will generate 1,000 barrels of oil per year forever. The extraction cost for this area is \$100/barrel. The dynamics are similar to those in problem 3: each year there is an 80% probability the oil price will not change and a 20% probability the oil price will change. From the upper or lower value, any oil price change will be to the middle, and from the middle the oil price change will be to the upper or lower value, each with 10% probability. The cash flows from oil extraction start one year after the \$1 million drilling cost is paid.

Assume that we choose whether to drill based on an expected present value calculation using an annual interest rate of 4%.

A. Write down an optimization problem to decide whether to drill.

- B. What are the state variables for this problem?
- C. Write down the Bellman equation and solve for the optimal policy. (Hint: one state variable is whether or not we have drilled already, and we cannot simply assert the value is 0 for either value of this variable. For solving the Bellman equation, start by solving for the value of a well that has already been drilled.) Do you drill when the price of oil is \$50/barrel? \$150/barrel? \$200/barrel?
- D. When the price of oil is \$150/barrel, drilling now gives us positive net present value but is not the right choice. Explain.
- E. (Extra for Experts) In the above analysis, we have assumed that once you have drilled the well, you extract the oil every period forever. How does the analysis change in the following three cases: (i) we have an option to abandon the mine at any time and save all current and future extraction costs (but forgo the corresponding oil revenues), (ii) we can start and stop production whenever we want (and therefore we produce only in years for which price exceeds extraction costs), and (iii) we can stop production any time but then we incur a cost  $\$C$  to restart production.
- F. (Challenger) Actual oil wells have a finite life. Also, we can choose the rate at which to extract oil, and the total amount of oil we can extract over the life of the well is decreasing in the rate of extraction. Formulate and solve for the optimal drilling policy in a model with these realistic features.