Problem Set 5 answers: Dynamic Programming
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At the start of class next week, submit only problem 4 for grading.

Problems 1 and 2 are relatively simple problems about dynamic programming, and something like these (with less calculation) might be on the final. Problems 3 and 4 are more involved problems that are similar to each other.

1. (short answer, no formulas) Suppose you want to trade on the discovery that the average movement in corn futures can be predicted by the last month's change in corn and soybean futures prices. Think about setting up a dynamic program to derive an optimal trading rule.

The answers are not unique!

A. What choice variable(s) will your problem have?

dynamic position in corn futures

B. What will be the objective function?

maybe expected terminal wealth, maybe a risk-averse utility function

C. What will be the constraints?

budget constraint (ideally reflecting trading costs), maybe limits on the position based on capital available

D. What are the state variables for your problem?

predicted corn futures price movement based on last month's change in corn and soybean futures, other variables (if any) determining the dynamics of the prediction, maybe wealth, maybe time (if a finite horizon problem), current position (if there are transaction costs)

2. (true-false) Dynamic programming concepts

A. The value function gives the value (under the optimal strategy) of the continuation of the problem.

TRUE

B. The maximum principle says the solution of the dynamic program also solves a one-period problem that maximizes the value that comes directly from today's price plus the indirect effect in the future as measured by the value function.

TRUE

C. Adding state variables is useful because state variables convert inequality constraints into equality constraints.

FALSE (Slack variables convert inequality constraints into equalities.)

D. Time should not be included as a state variable because it is not random.

FALSE (It can be one determinant of value. For example, in an investments problem the same wealth can have much different implications far from the horizon than near the horizon.)

3. When to terminate a business. Our large company owns a restaurant. Its annual cash inflows are random and can take on three values: \$1.8 million, \$1.2 million, and \$0.6 million, and the cash outflow is fixed and equal to \$1.3 million. Each year the cash inflow, whatever it is, stays the same with probability 0.8, or changes with probability 0.2. If the cash inflow this year is high or low (\$1.8 million or \$0.6 million), any move will be to the middle value (\$1.2 million). If the cash flow is at the middle value, it may move to the high value or the low value with probability 0.1 each.

At the beginning of each period, we have the option of closing the restaurant (permanently), in which case the value is zero because cash flows from that period onward are all zero. We want to maximize expected present value of remaining cash flows for all future dates, using a discount rate of 4%/year.

A. Write down the choice problem for when to close the restaurant.

Given c_0,
choose adapted closing indicator $x_t \in \{0,1\}$ to
maximize $E[\sum_{t=0}^{\infty} \frac{c_t}{1.04^t}(1-\sum_{s=0}^{t-1} x_s)]$, subject to
$\sum_{t=0}^{\infty} x_t \leq 1$ and
cash flow c_t can take on the values $(0.5, -0.1, -0.7)$ with dynamics given by a transition matrix

$$T = \begin{pmatrix} 0.8 & 0.2 & 0.0 \\ 0.1 & 0.8 & 0.1 \\ 0.0 & 0.2 & 0.8 \end{pmatrix},$$

where T_{ij} is the probability of having the jth cash flow next period if we have the ith cash flow this period. For example, $T_{12} = 0.2$ is the probability that $c_{t+1} = -0.1$ given that $c_t = 0.7$.

B. What are the state variables for this problem?

The cash flow c_t and whether we have closed $\sum_{s=0}^{t-1} x_s$. Since the value is always zero after closing (i.e., if the second state variable is 1), we will suppress this variable in the notation.

C. Write down the Bellman equation for the problem.

$$V(0.5) = \left(0.5 + \tfrac{1}{1.04}(0.8V(0.5) + 0.2V(-0.1))\right)^+$$
$$V(-0.1) = \left(-0.1 + \tfrac{1}{1.04}(0.1V(0.5) + 0.8V(-0.1) + 0.1V(-0.7))\right)^+$$
$$V(-0.7) = \left(-0.7 + \tfrac{1}{1.04}(0.2V(-0.1) + 0.8V(-0.7))\right)^+$$

If the "positive part" is relevant, that corresponds to closing the restaurant.

D. Solve the Bellman equation.

The optimal solution has $V(0.5) = 2.766$, $V(-0.1) = 0.719$, and $V(-0.7) = 0$, closing the restaurant only in the worst cash flow scenario. I derived this by solving the valuation equations under different assumptions about when the restaurant would be closed, noting that the optimum can't close the restaurant when cash flows are good and leave it open when cash flows are bad. For example, let $v = (V(0.5), V(-0.1), V(-0.7))$ and assume the restaurant is always left open. Then the Bellman equations are equivalent to the matrix equation

$$\begin{pmatrix} 1-\frac{0.8}{1.04} & -\frac{0.2}{1.04} & 0.0 \\ -\frac{0.1}{1.04} & 1-\frac{0.8}{1.04} & -\frac{0.1}{1.04} \\ 0.0 & -\frac{0.2}{1.04} & 1-\frac{0.8}{1.04} \end{pmatrix} v = \begin{pmatrix} 0.5 \\ -0.1 \\ -0.7 \end{pmatrix},$$

whose solution is $v = (0, -2.6, -5.2)$ which is obviously not consistent when we include the "postive part." Assuming the restaurant is closed only in the worst state, so $V(-0.7) = 0$, and letting $v = (V(0.5), V(-0.1))$, the Bellman equations become

$$\begin{pmatrix} 1-\frac{0.8}{1.04} & -\frac{0.2}{1.04} \\ -\frac{0.1}{1.04} & 1-\frac{0.8}{1.04} \end{pmatrix} v = \begin{pmatrix} 0.5 \\ -0.1 \end{pmatrix},$$

whose solution is $v = (2.766, 0.718)$, which is the true solution.