Problem set 1: Discussion

Financial Optimization

These are the same as problems 1 and 2 in problem set. However, I will use W_0=512. You should try it out for W_0=36.

Problem 1:

Let $Z = \frac{\pi^*}{\pi}$, Note that $E^*[Y]=E[ZY]$ (1)where Y is any random variable and E^* denotes expectation with respect to risk neutral probability.

(A) The optimization Problem then is : Choose W_T to maximize $E[U(W_T)]$ s.t $E^*[W_T]/R^T=W_0$ (1)

Then we have the Lagrangian

$$L \equiv E[U(W_T)] + \lambda[W_0 - E[ZW_T]/R^T] \tag{0.1}$$

(B) The Kuhn Tucker conditions for optimization are

$$\frac{\partial L}{\partial W_T} = 0 \tag{0.2}$$

Solving state by state we get FOC:

$$U^{'}(W_T) = \frac{Z\lambda}{R^T} \tag{0.3}$$

Hint: Note essentially we need to find $W_{T,S}$ in all states of the nature S at Terminal time. That is

$$E[U(W_T)] = \pi_{UU}U(W_{T,UU}) + \pi_{UD}U(W_{T,UD}) + \pi_{DU}U(W_{T,UDU}) + \pi_{DD}U(W_{T,DD})$$

Similarily we can open $E[ZW_T]$, we can then treat Each $W_{T,S}$ as a different choice variable and maximize.

$$\pi_{UU}U^{'}_{UU}(W_{T,UU}) = \frac{\pi_{UU}Z_{UU}\lambda}{R^T}$$

This is what we mean by state.

Note that W_T and Z are random variables and vary across states. Using the FOC and the budget constraint (1) we get the value of λ.

I will solve the problem for log utility. U should try it for the other form. Applying FOC (0.3) to log utility we get.

$$\frac{1}{W_T} = \frac{Z\lambda}{R^T} \tag{0.4}$$

Substituting the Value of W_T in (2) get $\lambda = \frac{1}{W_0}$, substituting back in (0.4) we get

$$W_T = \frac{R^T W_0}{Z} \tag{0.5}$$

(C) We can find the value of Z for each possible terminal state. Substituting it in 0.1 will give us the value of W_T in each possible terminal state. Since we have 3 time periods (t=0,1,2)the number of possible states $=2^{3-1}=4$. They are (U,U);(U,D);(D,U);(D,D)

Using the formulas given in the problem we get $\pi^*_U = \frac{1}{4}$, $\pi^*_D = \frac{3}{4}$, $\pi_U = \frac{1}{2}$, $\pi_D = \frac{1}{2}$, Now $\pi^*_{UU} = \frac{1}{16}$, $\pi^*_{UD} = \frac{3}{16}$, $\pi^*_{DU} = \frac{3}{16}$, $\pi^*_{DD} = \frac{9}{16}$, $\pi_{UU} = \pi_{UD} = \pi_{DU} = \pi_{DD} = \frac{1}{4}$

Using the formula for Z we get, $Z_{UU} = \frac{1}{4}$, $Z_{UD} = Z_{DU} = \frac{3}{4}$, $Z_{DD} = \frac{9}{4}$. Now using (0.5) and W_0=512, we get $W_{T,UU}$=3200, $W_{T,UD}$=$W_{T,DU}$=1066.7, $W_{T,DD}$=355.6. Note we still need to verify second order conditions. It is satisfied for the log utility.

We can now easily calculate the maximum expected utility of terminal consumption.

(D) To compute the dynamic portfolio strategy. Let $W_{1,U}$, $W_{1,D}$be the wealth at t=1, in states U and D respectively. Let $\alpha_{1,U}$ be the portfolio weight for the risky asset. Then One approach is:

$$\alpha_{1,U} W_{1,U} U + (1 - \alpha_{1,U}) W_{1,U} R = 3200 \tag{0.6}$$

$$\alpha_{1,U}W_{1,U}D + (1-\alpha_{1,U})W_{1,U}R = 1066.7 \tag{0.7}$$

Substracting (0.7) from (0.6) we get

$$\alpha_{1,U}W_{1,U} \approx 2134 \tag{0.8}$$

Using (0.6) and 0.8 we get $W_{1,U}$≈1277.8$\alpha_{1,U}$≈1.7, $W_{1,D}$≈427Similarily we can find $\alpha_{1,D}$ and α_0.

First note that $\alpha_{1,U}$=$\alpha_{1,D}$=α_0≈1.7. That is the proportion of wealth invested in the risky asset is constant. Secondly, notice that the investor prefers to go long in the risky asset and shorts the riskless asset.

Alternatively we could find $W_{1,U}$ using risk neutral Valuation.

$$W_{1,U} = (\pi^*_U W_{T,UU} + \pi^*_D W_{T,UD})/R$$

This aslo gives the same result.

Problem 2:

The only change from the previous question is that there is now additional constraint on terminal consumption.

(A) The optimization Problem then is : Choose W_T to maximize $E[U(W_T)]$ s.t $E^*[W_T]/R^T$=W_0 (1) and $W_{T,S} \geqq W_0$ (2) where S=UU,UD,DU,DD

Then we have the Lagrangian

$$L \equiv E[U(W_T)] + \lambda[W_0 - E[ZW_T]/R^T] + \sum_S \gamma_S(W_0 - W_{T,S}) \tag{0.1}$$

(B) The Kuhn Tucker conditions for optimization are:

$$\frac{\partial L}{\partial W_T} = 0 \tag{0.2}$$

Opening the Expectation operator we get

$$\pi_s U^{'}(W_{T,S}) = \frac{\pi_s Z_s \lambda}{R^T} + \gamma_s \tag{0.3}$$

Also,

$$\gamma_S(W_0 - W_{T,S}) = 0, \gamma_s \geq 0 \tag{0.4}$$

and

Using the FOC and the budget constraint (1) we get the value of $\lambda$and γ.

(C)As above we can find the value of Z in all possible states. Given the solution to the above problem, I start with the conjecture that (2) is not binding for s=UU,UD,DU i.e. ($\gamma_s = 0$)and is binding for DD. It is straightfoward to see that it cannot be the case that (2) is not binding for all s, otherwise we would have the same solution as above. However, the above solution does not satisfy (2) for s=DD.

Using FOC we get,

$$\frac{1}{W_{T,s}} = \frac{Z_s \lambda}{R^T} \tag{0.5}$$

and

$$\frac{\pi_{DD}}{W_{T,DD}} = \frac{Z_{DD} \lambda}{R^T} + \gamma_{DD} \tag{0.6}$$

Now since we have assumed (2) is binding for S=DD, we have $W_{T,DD}$=W_0 (3). Using (3) and (0.6) we get the value of γ_{DD}.

Substituting the values of $W_{T,s}$ from (0.5) and (3) in budget constraint 1, we get the value of λ.

$$\lambda = [\frac{W_0(1 - \pi_{DD} Z_{DD} R^{-T})}{\pi_{UU} + \pi_{UD} + \pi_{DU}}]^{-1} \approx \frac{1}{437}$$

Now we can proceed as in problem 1 and find $W_{T,S}$ for all S (Substitute λ in (0.5)). $W_{T,UU}$=2731, $W_{T,UD}$=$W_{T,DU}$=910, $W_{T,DD}$=512.

(D) We can solve for portfolio weights α as in problem 1. However not that now α is both time and state dependent. We get $\alpha_{1,U}$≈1.7, $\alpha_{1,D}$≈0.8, α_0≈1.2.

Intution for time varying portfolio weights: This is due to the additional consumption constraint. This can be thought of as akin to some kind of margin requirements. The investor switches to bond if stock prices fall. However, stock prices increase, the investor moves back to the unconstrained portfolio weight of 1.6.