FIN 550 Exam Bonus Problem Answer

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5. Bonus problem (30 bonus points)

Consider the Markov switching model in problem 3 and in particular the project described in part 3F. Suppose that instead of starting the project immediately, we can pay $\$3,000$ per unit time to keep the project alive without undertaking it. To get any credit for this problem, you must show your work, and there is no use just guessing.

A. What is the optimal strategy? You can restrict your analysis to the only sensible strategies (a) abandon the project, (b) invest immediately, and (c) spend $3,000$ per unit time to keep the project alive until we arrive in state 1, and at that time invest in the project.

Alternative (a) has NPV$=0$ so it is dominated by alternative (b) which has NPV$=10,000$ (from the calculation in part 3E). If instead we delay but keep the project alive so long as we are in state 2, during that time we pay $3,000$ per unit time but we can also collect $100,000 \times 5\% = \$5,000$ per unit time in interest on the $\$100,000$ that hasn't been invested yet, for a net cash flow of $1,000$ per unit time, which is larger than the $1,000$ per unit time we would have gotten if we invested at time zero. Therefore, it pays to wait but pay to keep the investment alive.

B. What is the NPV of the optimal strategy?

Since the net increase in cash flow is $\$1,000$ per unit time, the increase in value over investing immediately is the expected present value of receiving $\$1,000$ per unit time until the first time we enter state 1. Let $P(t)$ be the probability we haven't entered state 1 at or before time t. Then $P'(t) = -.15P(t)$ and $P(0) = 1$, so that $P(t) = \exp(-.15t)$. Therefore, the increase of expected NPV compared to investing immediately is

$$\int_{t=0}^{\infty} 1000e^{-.05t}e^{-.15t}dt \quad = \quad \frac{1000}{.20}$$

$$= \quad \$5,000$$

So, the NPV from keeping the project alive until state 1 is $\$10,000+5,000 = \$15,000$.

A second solution

I like the economic argument above because it is easy to understand and economizes on calculations. To confirm the answer, I checked my solution using a alternative mathematical argument. First, we can derive as the expected NPV, priced at time τ, of starting the project at time τ. This value does not depend on τ since this is a Markov model that looks the same going forward at every time so long as the state variables are the same, and at time τ we are always in state 1 no matter what the realized value of τ is. This is the same as the calculation in problem 3 except that we have the initial condition $\pi(0) = (1,0)^T$ instead of $\pi(0) = (0,1)^T$. As before, define $c = (10,1)^T$ to be the vector of costs across states, in thousands. Then we have

$$\begin{aligned}
E[PV] &= \int_{t=0}^{\infty} \left(c^T \left(\begin{bmatrix} .6 \\ .4 \end{bmatrix} + \begin{bmatrix} .4 \\ -.4 \end{bmatrix} e^{-.25t} \right) \right) e^{-.05t} dt \\
&= \frac{6.4}{.05} + \frac{3.6}{.30} \\
&= 128 + 12 = 140.
\end{aligned}$$

so the NPV in thousands looking forward at time τ is $140 - 100 = 40$. So, conditional on τ, the NPV in thousands priced at time 0 is

$$\begin{aligned}
& 40e^{-.05\tau} - \int_{t=0}^{\tau} 3e^{-.05t} dt = 40e^{-.05\tau} - \frac{3}{.05}(1 - e^{-.05\tau}) \\
= \quad & 100e^{-.05\tau} - 60.
\end{aligned}$$

Since we do not know τ, we have to take an expectation over its possible realizations. The cumulative distribution function $F(t)$ for the first time τ we enter state 1 is by definition the probability that $\tau \leq t$, which is one minus the probability that $\tau > t$, which is $P(t)$. So, $F(\tau) = 1 - P(\tau) = 1 - \exp(-.15\tau)$ and therefore the probability density is $f(\tau) = F'(\tau) = .15\exp(-.15\tau)$. Therefore, the expected NPV for the waiting strategy is

$$\begin{aligned}\int_{\tau=0}^{\infty}(100e^{-.05\tau} - 60).15e^{-.15\tau}d\tau &= \frac{.15 * 100}{.2} - \frac{.15 * 60}{.15} \\ &= 75 - 60 \\ &= 15,\end{aligned}$$

which is \$15,000, the same as above. Therefore, paying to wait until state 1 arrives is optimal because it has a higher present value than both investing immediately (NPV= \$10,000) and abandoning the project (NPV= \$0).

A third solution

Without providing many details, I would like to note a third approach for evaluating the value of paying to keep the project alive until state 1 arrives when we invest. For this, we want to distinguish whether 1 has or has not ever occured, and we will consider the probability vector $p(t)$ satisfying $p'(t) = Bp(t)$ where

$$p(t) = \begin{pmatrix} -.1 & .15 & .15 \\ 0 & -.15 & 0 \\ .1 & 0 & -.15 \end{pmatrix}$$

and the states in order are "state 1," "state 2 and never before in state 1," and "state 2 and previously visited state 1. Doing the usual analysis using eigenvalues and eigenvectors of B and imposing the initial condition that $p(0) = (0,1,0)^T$, we have that

$$p(t) = \begin{bmatrix} .6 \\ 0 \\ .4 \end{bmatrix} + e^{-.15t}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + e^{-.25t}\begin{bmatrix} -.6 \\ 0 \\ .6 \end{bmatrix}.$$

Using the state-dependent cash flow in thousands $C = (10, -3, 1)^T$ (note this cash flow is not a difference like the cash flow of \$1,000 in the first solution and therefore also does not include interest on the deferred investment), we can write the NPV as

$$
\begin{aligned}
\text{NPV} &= \int_{t=0}^{\infty}(-100{p_2}'(t) + C^T p(t))e^{-rt}dt \\
&= \int_{t=0}^{\infty}\left(-100 \times .15e^{-.15t} + C^T\left(\begin{bmatrix} .6 \\ 0 \\ .4 \end{bmatrix} + e^{-.15t}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right.\right. \\
&\qquad\quad \left.\left. + e^{-.25t}\begin{bmatrix} -.6 \\ 0 \\ .6 \end{bmatrix}\right)\right) e^{-.05t}dt \\
&= \frac{-15}{.2} + \frac{6.4}{.05} + \frac{-4}{.2} + \frac{-5.4}{.3} \\
&= -75 + 128 - 20 - 18 \\
&= 15.
\end{aligned}
$$

Note that this analysis using B can also be used to solve problem 3, using the cost vector $c = (6, 1, 1)^T$, so with this approach is not necessary to do two different eigenvalue analyses for problems 3 and 5.