FIN 550 Exam answers

Phil Dybvig
December 13, 2012

1. True-False (25 points)

A. Every unconstrained problem has at least one interior solution.

False. (An unconstrained problem may not have any solution at all. For example consider the problem

Choose x to
maximize x^2

which increases without bound as $|x|$ increases.)

B. In a constrained problem with a concave objective function, a local maximum is always a global maximum.

False. (This would be true if the constraint set were known to be convex, but not in general. Consider the problem

Choose x_1 and x_2 to
minimize $(x_1 - 1)^2 + (x_2 - 2)^2$
subject to $\max(x_1, x_2) \geq 0$

In this problem, $(x_1, x_2) = (1, 0)$ and $(x_2, x_2) = (0, 2)$ are both local optima, but only $(1, 0)$ is a global optimum.)

C. Skewness of a random variable is always positive.

False. (A random variable that equals 1 with probability $2/3$ and -2 with probability $1/3$ is negatively skewed.)

D. The sum of the eigenvalues of a negative semidefinite matrix is always negative.

False. (It could be zero, as it is for every square matrix whose entries are all zero.)

E. A linear program with a bounded primal always has an optimal solution.

False. (Consider the program

Choose x to
maximize x
subject to $x \geq 3$ and $x \leq 2$

This program is bounded but has no feasible solution and therefore no optimal solution.)

2. Linear Programming (25 points) Consider the following linear program:

Choose nonnegative x_1, x_2, x_3, and x_4 to
maximize $x_1 + 6x_2 + 4x_3 + 4x_4$, subject to
$x_1 + 3x_2 + 3x_3 + 5x_4 \leq 6$ and
$x_1 + x_2 \leq 3$

A. What is the dual linear program?

Choose nonnegative y_1 and y_2 to
minimize $6y_1 + 3y_2$, subject to
$y_1 + y_2 \geq 1$
$3y_1 + y_2 \geq 6$
$3y_1 \geq 4$ and
$5y_1 \geq 4$

B. Solve the dual problem.

Graphing the problem shows that the only extreme points in the feasible set are $(y_1, y_2) = (2, 0)$ (which has equality in the second constraint and

$y_2 \geq 0$) and $(y_1, y_2) = (4/3, 2)$ (which has equality in the second and third constraints). These two points have values 12 and 14, respectively, so the first point $(2, 0)$ is preferred (because we are minimizing). Also, this is a global minimum because the objective function is increasing in both y_1 and y_2 so going away in feasible extreme directions increases the objective function.

C. Use the solution to the dual problem to solve the primal problem.

Since only the second constraint is binding and because the dual variables of the primal are the choice variables of the dual and vice versa, complementary slackness imples that $x_1 = x_3 = x_4 = 0$ and $x_2 \geq 0$ and the first constraint of the primal problem holds with equality. Therefore, $3x_2 = 6$ or $x_2 = 2$ and therefore $(x_1, x_2, x_3, x_4) = (0, 2, 0, 0)$ is the optimal solution of the primal problem.

D. Show that the strong duality theorem holds in this example.

The optimal solution of the primal has value $(1, 6, 4, 4) \cdot (0, 2, 0, 0) = 12$, and the optimal solution of the dual has value $(6, 3) \cdot (2, 0) = 12$. Since the two are the same, strong duality holds.

3. Kuhn-Tucker Conditions (25 points)

Consider the following optimization problem:

Choose c_u and c_d to
maximize $\frac{2}{3}\log(c_u - 3) + \frac{1}{3}\log(c_d - 3)$, subject to
$\frac{4}{5}(\frac{1}{2}c_u + \frac{1}{2}c_d) = 6$,
$c_d \geq 6$,
and
$c_u \geq 6$.

This is a single-period choice of investment for consumption in a binomial model with translated log utility, initial wealth of 6, actual probabilities 2/3 and 1/3, risk-neutral probabilities 1/2 and 1/2, and riskfree rate of 25% (and therefore discount factor 4/5).

A. What are the objective function, choice variables, and constraints?

objective function: $\frac{2}{3}\log(c_u - 3) + \frac{1}{3}\log(c_d - 3)$

choice variables: c_u and c_d

constraints: $\frac{4}{5}(\frac{1}{2}c_u + \frac{1}{2}c_d) = 6$, $c_d \geq 6$, and $c_u \geq 6$

B. What are the Kuhn-Tucker conditions?

Now, $f(c_u, c_d) = \frac{2}{3}\log(c_u - 3) + \frac{1}{3}\log(c_d - 3)$, so $\nabla f(c_u, c_d) = (2/(3(c_u - 3)), 1/(3(c_d - 3)))^T$. The first constraint is an equality constraint with function $g(c_u, c_d) = (2/5)(c_u + c_d) - 6$ and there are two inequality constraints with $g_1(c_u, c_d) = 6 - c_u$ and $g_2(c_u, c_d) = 6 - c_d$ (since inequality constraints for maximizations are written with $\leq$ to make the direction into the feasible set correct when the Lagrangian multipliers are positive). The corresponding gradients for the constraints are $\nabla g(c_u, c_d) = (2/5, 2/5)$, $\nabla g_1(c_u, c_d) = (-1, 0)$, and $\nabla g_2(c_u, c_d) = (0, -1)$. Therefore, the K-T conditions are

$$\begin{pmatrix} \frac{2}{3(c_u-3)} \\ \frac{1}{3(c_d-3)} \end{pmatrix} = \lambda \begin{pmatrix} \frac{2}{5} \\ \frac{2}{5} \end{pmatrix} + \lambda_u \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \lambda_d \begin{pmatrix} 0 \\ -1 \end{pmatrix} \qquad (*)$$

$\lambda_u, \lambda_d \geq 0$

$\lambda_u(6 - c_u) = 0$

$\lambda_d(6 - c_d) = 0$

Optionally, include the constraints:

$\frac{4}{5}(\frac{1}{2}c_u + \frac{1}{2}c_d) = 6$

$c_d \geq 6$

$c_u \geq 6$

C. Solve the optimization problem.

Note that $\lambda > 0$ since ∇f has all positive elements and the nonnegative multipliers λ_u and and λ_d multiply nonpositive vectors in the main first-order condition $(*)$. Therefore, we can take $\lambda > 0$ and consider the 4 cases determined by $\lambda_u = 0$ or not and $\lambda_d = 0$ or not. When solving the problem, we can stop when we find a solution to the first-order condition. Because the objective function is smooth (it is C^∞ on its domain) and concave[1] and the constraints are affine and nondegenerate,[2] the first-order conditions are necessary and sufficient.

$\lambda_u = \lambda_d = 0$:

In this case, the f.o.c. (*) is

$$\begin{pmatrix} \frac{2}{3(c_u-3)} \\ \frac{1}{3(c_d-3)} \end{pmatrix} = \lambda \begin{pmatrix} \frac{2}{5} \\ \frac{2}{5} \end{pmatrix}.$$

Substituting in the budget constraint $\frac{4}{5}(\frac{1}{2}c_u + \frac{1}{2}c_d) = 6$, we solve for $(c_u, c_d) = (9, 6)$ which is a solution since it satisfies the K-T conditions with ($\lambda = 5/9$ and $\lambda_u = \lambda_d = 0$) and all the constraints of the problem. If we checked this case first, we would be done.

$\lambda_u > 0$ and $\lambda_d = 0$:

In this case, complementary slackness implies $c_u = 6$ so the budget constraint $\frac{4}{5}(\frac{1}{2}c_u + \frac{1}{2}c_d) = 6$ implies $c_d = 9$. The f.o.c. (*) in this case is

$$\begin{pmatrix} \frac{2}{3(c_u-3)} \\ \frac{1}{3(c_d-3)} \end{pmatrix} = \lambda \begin{pmatrix} \frac{2}{5} \\ \frac{2}{5} \end{pmatrix} + \lambda_u \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

so we have $\lambda = 5/36$ and $\lambda_u = -1/6$. This cannot be a solution because the

[1]The second derivative of the objective function is zero off the diagonal and negative on the diagonal. Therefore, it is negative definite since the two eigenvalues are the diagonal entries.

[2]Since the three constraints are never all satisfied with equality at the same time, we only need to observe that each pair of constraint gradients is linearly independent.

sign on λ_u is wrong.

$\lambda_u = 0$ and $\lambda_d > 0$:

In this case, complementary slackness implies $c_d = 6$ so the budget constraint implies $c_u = 9$. The f.o.c. (*) in this case is

$$\begin{pmatrix} \frac{2}{3(c_u-3)} \\ \frac{1}{3(c_d-3)} \end{pmatrix} = \lambda \begin{pmatrix} \frac{2}{5} \\ \frac{2}{5} \end{pmatrix} + \lambda_d \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

so we have $\lambda = 5/9$ and $\lambda_d = 0$. Although this does not satisfy our assumption that $\lambda_d > 0$, this solution does satisfy the constraints and the K-T conditions (it is the same solution we found above assuming $\lambda_u = \lambda_d = 0$), so if we found this first, we would be done.

$\lambda_u, \lambda_d > 0$:

In this case, all three constraints would have to be binding. However, if $c_u = c_d = 6$ the budget constraint $\frac{4}{5}(\frac{1}{2}c_u + \frac{1}{2}c_d) = 6$ is not satisfied, so this case cannot happen.

The solution was the one in the first case (also found in the third case) with $c_u = 9$, $c_d = 6$, $\lambda = 5/9$, $\lambda_u = 0$, and $\lambda_d = 0$.

4. REGIME-SWITCHING (25 points) Consider a two-state Markov Chain in continuous time. Regime switches take place at the following rates:

state 1 $\rightarrow$ state 2	probability 0.1/year
state 2 $\rightarrow$ state 1	probability 0.1/year

Initially (at time $t = 0$), we are in state 1.

A. What is the matrix A in the ODE

$$\pi'(t) = A\pi(t)$$

describing the dynamics of the vector $\pi(t)$ of future regime probabilities?

$$A = \begin{pmatrix} -0.1 & 0.1 \\ 0.1 & -0.1 \end{pmatrix}$$

B. Solve for the eigenvalues of A.

$$\det(A - \lambda I) = det \begin{pmatrix} -0.1 - \lambda & 0.1 \\ 0.1 & -0.1 - \lambda \end{pmatrix} = 0$$

$$\lambda^2 + .2\lambda + .01 - .01 = 0$$

$$\lambda = 0, -.2$$

C. Solve for the associated eigenvectors.

$\lambda = 0$:

$$(A - \lambda I)x = \begin{pmatrix} -0.1 & 0.1 \\ 0.1 & -0.1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Try the normalization $x_2 = 1$. Then we have $x_1 = 1$ and the eigenvalue is $(1, 1)^T$.

$\lambda = -.2$:

$$(A - \lambda I)x = \begin{pmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Try the normalization $x_2 = 1$. Then we have $x_1 = -1$ and the eigenvalue is $(-1, 1)^T$.

D. Write down the general solution of the ODE.

$$\pi(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-.2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

E. Write down the particular solution corresponding to the initial condition that we start in state 1.

Substituting $\pi(0) = (1,0)^T$, we find $c_1 = 1/2$ and $c_2 = -1/2$ so we have

$$\pi(t) = \tfrac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \tfrac{1}{2}e^{-.2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} 1+e^{-.2t} \\ 1-e^{-.2t} \end{pmatrix}.$$

F. A project costing \$45,000 has a cash flow of \$10,000/year in state 1 but loses money with cash flow −\$8,000/year in state 2. The cash flows continue forever. If the continuously-compounded interest rate is 10%/year, does this project have a positive expected NPV?

In thousands, we have

$$\begin{aligned} NPV &= \int_{t=0}^{\infty} e^{-.1t}(10,-8)\pi(t)dt \\ &= \int_{t=0}^{\infty} e^{-.1t}(10\frac{1+e^{-.1t}}{2} - 8\frac{1-e^{-.1t}}{2})dt \\ &= \int_{t=0}^{\infty}(e^{-.1t} + 9e^{-.2t})dt \\ &= \frac{1}{.1} + \frac{9}{.3} \\ &= 40. \end{aligned}$$

Therefore, the NPV is $40,000 - 45,000 = -5,000 < 0$. It is not positive.

5. Bonus problem (30 bonus points)

Consider the Markov switching model in Problem 4 and in particular the project described in part 4E. Suppose that instead of continuing forever, the project can be costlessly (and irreversibly) abandoned anytime you want, without any cash flows at that time or subsequently. To get any credit for this problem, you must show your work, and there is no use just guessing.

A. What is the optimal strategy? You can restrict your analysis to the only sensible strategies (a) do nothing, (b) buy the project and never abandon it, and (c) buy the project and continue to operate it until state 2 occurs, at which time you abandon the project.

Obviously, the NPV of (a) is zero and the NPV of (b) was computed in problem 4 as $-\$5,000$. There are several ways of thinking about the NPV for (c); the key is that we have to think about the dynamics of the state transitions differently so we distinguish states in which we have previously visited state 2 (and abandoned the project) from states in which we have not. We could define the states by (i) in state 1 and never before in state 2, (ii) in state 2, and (iii) in state 1 but visited state 2 before. However, once we visit state 2, the project is over and we don't care whether we are in state 1 or state 2, so we can simply define the states as (I) haven't visited state 2 yet and (II) have visited state 2. Using these state definitions, we have that $\pi'(t) = A\pi(t)$ where

$$A = \begin{pmatrix} -0.1 & 0 \\ 0.1 & 0 \end{pmatrix}.$$

Using calculations as in problem 4, it is easy to compute that A has eigenvalues 0 and $-.1$ with corresponding eigenvectors $x^1 = (0,1)^T$ and $x^2(1,-1)^T$. So, the general solution of $\pi'(t) = A\pi(t)$ is given by $\pi(t) = c_1(0,1)^T + e^{-.1t}(1,-1)^T$. The particular solution satisfying $\pi(0) = (1,0)^T$ is

$$\pi(t) = \tfrac{1}{2}\begin{pmatrix} e^{-.1t} \\ 1 - e^{-.1t} \end{pmatrix}.$$

The present value in thousands of starting the project at the outset and abandoning it once state 2 occurs is:

$$\begin{aligned} PV &= \int_{t=0}^{\infty} e^{-.1t} 10 e^{-.1t} dt \\ &= \frac{10}{.2} \\ &= 50 \end{aligned}$$

So, the NPV of (c) is $\$50,000 - \$45,000 = \$5,000$. Since this NPV is highest, it is the optimal strategy.

B. What is the NPV of the optimal strategy?

The NPV of (c), which is $\$5,000$.