Problem Set 3 Answers to Problems 1 and 3
Linear Programming Duality and Fundamental Theorem of Asset Pricing
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1. Dual LP

A. Write down the dual LP for the following LP

The primal LP is:

Choose nonnegative $x_1$, $x_2$, and $x_3$ to
minimize $x_1 - 2x_2 + 3x_3$, subject to
$x_1 \geq 1$ and
$x_1 + 2x_2 + 4x_3 \geq 2$.

The dual LP is:

Choose nonnegative $y_1$ and $y_2$ to
maximize $y_1 + 2y_2$, subject to
$y_1 + y_2 \leq 1$,
$2y_2 \leq -2$, and
$4y_2 \leq 3$.

B. For the primal and dual LP, answer the following. Is the LP feasible? Is the LP bounded? If the LP has a solution, what is it?

primal:
... feasible? yes $x = (1, 1, 1)$ is feasible
... bounded? no $\Delta = (0, 1, 0) \geq 0$ reduces the objective and does not make the constraints more binding ($\Delta^T(1, 0, 0)^T \geq 0$ and $\Delta^T(1, 2, 4)^T \geq 0$)
... exist a solution? no (adding $(0, 1, 0)$ to any candidate solution is still feasible and increases value)

dual:
... feasible? no, since $y_2 \geq 0$, $2y_2 \not\leq -2$
... bounded? yes, any direction $\Delta \geq 0$ that does not make the constraints more binding has $\Delta_1 + \Delta_2 \leq 0$, $2\Delta_2 \leq 0$, (and $4\Delta_2 \leq 0$, a fact we will not need) and therefore the sum $(\Delta_1 + \Delta_2) + (2\Delta_2)/2 = \Delta_1 + 2\Delta_2 \leq 0$.

Consequently, the objective function cannot be improved: $\Delta_1 + 3\Delta_2 \not> 0$.
... exist a solution? no, since it is not feasible

C. Discuss the results in terms of the theorem about feasibility and boundedness of the primal and dual.

The theorem says the primal is feasible if and only if the dual is bounded, and the primal is bounded if and only if the dual is feasible. In our case this is consistent: the primal is feasible and the dual is bounded, while the primal is unbounded and the dual is infeasible.

3. Finding arbitrage: puts

You can buy HAL stock for \$59.36/share or go short at \$59.32 and you can also buy or sell listed puts all maturing on the same date in August (these are all-in prices based on most recent trades plus an estimate of half the spread plus trading costs):

| strike | put ask | put bid |
|---|---|---|
| 45 | 1.52 | 1.48 |
| 50 | 2.77 | 2.73 |
| 55 | 2.99 | 2.85 |
| 60 | 5.95 | 5.85 |
| 65 | 10.65 | 10.35 |

In addition, the riskless borrowing rate for the maturity of the options is 1% simple interest and the lending rate is 0.5% simple interest.

A. Set up the state-space tableau for short and long positions in the put options and the underlying stock and riskfree borrowing/lending. Calculate the payoffs based on terminal stock prices 0, 45, 50, 55, 60, 65, and 1000 (all the strike prices plus two extreme values).

The state-space tableau for payoffs at maturity for long positions only is

$$X = \begin{pmatrix} 45 & 50 & 55 & 60 & 65 & 0 & 100 \\ 0 & 5 & 10 & 15 & 20 & 45 & 100 \\ 0 & 0 & 5 & 10 & 15 & 50 & 100 \\ 0 & 0 & 0 & 5 & 10 & 55 & 100 \\ 0 & 0 & 0 & 0 & 5 & 60 & 100 \\ 0 & 0 & 0 & 0 & 0 & 65 & 100 \\ 0 & 0 & 0 & 0 & 0 & 1000 & 100 \end{pmatrix}$$

and the vector of bid prices is $p_b^\top = (1.48, 2.73, 2.85, 5.85, 10.35, 59.32, 99.099)$, and the vector of ask prices is $p_a^\top = (1.52, 2.77, 2.99, 5.95, 10.65, 59.36, 99.50249)$, where we computed $99.099 = 100/1.01$ and $99.50249 = 100/1.005$.

Therefore, the full state-space tableau for cash flows initially and at maturity with short and long positions separately is given by

$$Z = \begin{pmatrix} -p_a^\top & p_b^\top \\ X & -X \end{pmatrix}.$$

For this analysis, assume these are European options.

B. Write down a linear programming problem that searches for an arbitrage given these trading opportunities.

Choose $\eta$ to
maximize $\mathbf{1}^\top Z\eta$, subject to
$Z\eta \geq 0$
$\mathbf{1}^\top Z\eta \leq 1$.

(Other LPs are possible.)

C. Use a computer to search for an arbitrage opportunity.

When I solved this in another class where I covered the material, I used Solver in Excel. I entered the matrix Z in the transpose (with rows and columns reversed), since more colums fit on the screen than rows in Excel. I did not compute $\mathbf{1}^\top Z$ in advance, although that might run faster: this problem is small enough so it doesn't matter. Importantly, I told Solver this is an LP, so it knew not to use an algorithm that tries to approximate and invert the second derivative of the objective. The optimal solution found by Solver buys .168067 each of the puts with strike 45 and 55 and sells .336134 of the put with strike 50.

(This solution is not unique.)

D. Examine the numerical solution and describe it qualitatively.

The strategy is a butterfly spread and pays off up front and when the stock price ends near 50 but pays zero when the stock price ends below 45 or above 50. In practice, we would have to confirm we can trade at the prices. It is likely that the furthest out-of-the-money option does not trade much and the "most recent" trade's price may not be current.

(If a different solution is found, the discussion would be different. This is not the only good discussion of this solution.)