

Problem Set 5: Eigenvalues, eigenvectors, and regime-switching models
FIN 550: Numerical Methods and Optimization in Finance
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This homework is for your benefit for preparing for the test. Answers to Problems 1 and 3 are provided. There is also an Extra for Experts problem (Problem 2) for your enjoyment if you need a diversion. There is nothing to hand in on this assignment.

1. Consider the matrix

$$D = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}$$

A. Compute the eigenvalues λ_1 and λ_2 of A .

B. Compute corresponding eigenvectors.

C. Let $x_0 = (3, 2)^T$. Write x_0 as a linear combination of the eigenvectors.

D. Use the eigenvalues and eigenvectors to compute $A^5 x_0$.

2. Extra for Experts (an optional problem for students of superior ability, preparation, or ambition) Prove your answers.

Let

$$F = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \text{ and } g = \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

(a) Then, how many digits are there in the largest element of

$$y = F^{1000001} g,$$

represented as a number in base 8?

(b) What is $2y_2 - 3y_1$?

2. Consider a model with three economic scenarios: (1) healthy economy, (2) recession, and (3) depression. These states are assumed to follow a Markov switching model in continuous time. From a healthy economy, the economy has a probability per unit time of .05 of moving to a recession but cannot move

directly to a depression. From a recession, the economy has a probability per unit time of .03 of moving to a healthy economy and a probability per unit time of .02 of moving to a depression. From a depression, the economy has a probability per unit time of .05 of moving to a recession but cannot move directly to a healthy economy.

A. Let $\pi(t) = (\pi_1(t), \pi_2(t), \pi_3(t))^T$ be the vector of the probabilities of the three states at a future time t given the information now. Write down a first-order vector ODE satisfied by $\pi(t)$.

B. Find the general solution of the vector ODE given in part A.

C. Find the solution of the ODE that satisfies the initial condition that we are in a recession at time $t = 0$.

D. We have a possible investment project that requires an initial investment of \$100,000. The project pays a cash flow c_t of \$7,000/year when the economy is healthy, \$1,000/year in a recession, and \$0/year in a depression. If the interest rate is 2%, is the expected net present value

$$\int_{t=0}^{\infty} e^{-rt} E[c_t] dt - 100,000$$

of the cash flows positive?