Problem Set 5 Answers: Eigenvalues, eigenvectors, and regime-switching models
FIN 550: Numerical Methods and Optimization in Finance
P. Dybvig

1. Consider the matrix

$$D = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}$$

A. Compute the eigenvalues λ_1 and λ_2 of A.

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} -\lambda & 2 \\ -1 & 3-\lambda \end{pmatrix} \\ &= -\lambda(3-\lambda) - 2(-1) \\ &= 2 - 3\lambda + \lambda^2 \\ &= (\lambda - 2)(\lambda - 1). \end{aligned}$$

Therefore, the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 1$. (The ordering is arbitrary, so saying $\lambda_1 = 1$ and $\lambda_2 = 2$ would also be correct.)

B. Compute corresponding eigenvectors.

for $\lambda_1 = 2$, we have $(D - \lambda_1 I)x = 0$ or

$$\begin{pmatrix} 0-2 & 2 \\ -1 & 3-2 \end{pmatrix} x = 0.$$

The first row tells us that $-2x_1 + 2x_2 = 0$ or $x_1 = x_2$ (and the second row tells us the same). Arbitrarily setting $x_2 = 1$ (which corresponds to choice of scaling), we have that the first eigenvector can be taken to be

$$x^1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We can confirm this by checking the eigenvalue equation $Dx = \lambda x$:

$$\begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \times 1 + 2 \times 1 \\ -1 \times 1 + 3 \times 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For the second eigenvalue $\lambda_2 = 1$, we have $(D - \lambda_2 I)x = 0$ or

$$\begin{pmatrix} 0-1 & 2 \\ -1 & 3-1 \end{pmatrix} x = 0.$$

The first row tells us that $-x_1 + 2x_2 = 0$ or $x_1 = 2x_2$ (and the second row tells us the same). Arbitrarily setting $x_2 = 1$ (which corresponds to choice of scaling), we have that the second eigenvector can be taken to be

$$x^1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

We can confirm this by checking the eigenvalue equation $Dx = \lambda x$:

$$\begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \times 2 + 2 \times 1 \\ -1 \times 2 + 3 \times 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 1 \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

C. Let $x_0 = (3, 2)^T$. Write x_0 as a linear combination of the eigenvectors.

Let $x_0 = c_1 x^1 + c_2 x^2$. Equating the transpose of each side we have $c_1(1, 1) + c_2(2, 1) = (3, 2)$, or

$$\begin{aligned} c_1 + 2c_2 &= 3 \\ c_1 + c_2 &= 2 \end{aligned}$$

Taking the difference of the two equations, we have that $c_2 = 1$ and therefore from either equation we have $c_1 = 1$. So, $x_0 = x^1 + x^2$.

D. Use the eigenvalues and eigenvectors to compute $A^5 x_0$.

$$A^5 x_0 = A^5(x^1 + x^2) = A^5 x^1 + A^5 x^2 = {\lambda_1}^5 x^1 + {\lambda_2}^5 x^2 = 32x^1 + x^2 = \begin{pmatrix} 34 \\ 33 \end{pmatrix}.$$

3. Consider a model with three economic scenarios: (1) healthy economy, (2) recession, and (3) depression. These states are assumed to follow a Markov

switching model in continuous time. From a healthy economy, the economy has a probability per unit time of .05 of moving to a recession but cannot move directly to a depression. From a recession, the economy has a probability per unit time of .03 of moving to a healthy economy and a probability per unit time of .02 of moving to a depression. From a depression, the economy has a probability per unit time of .05 of moving to a recession but cannot move directly to a healthy economy.

A. Let $\pi(t) = (\pi_1(t), \pi_2(t), \pi_3(t))^T$ be the vector of the probabilities of the three states at a future time t given the information now. Write down a first-order vector ODE satisfied by $\pi(t)$.

$$\pi'(t) = A\pi(t)$$

where

$$A = \begin{pmatrix} -0.05 & 0.03 & 0 \\ 0.05 & -0.05 & 0.05 \\ 0 & 0.02 & -0.05 \end{pmatrix}$$

B. Find the general solution of the vector ODE given in part A.

First, find the eigenvalues (computing the determinant by expanding around the first column):

$$\begin{aligned} 0 = \det(A - \lambda I) &= det \begin{pmatrix} -0.05-\lambda & 0.03 & 0 \\ 0.05 & -0.05-\lambda & 0.05 \\ 0 & 0.02 & -0.05-\lambda \end{pmatrix} \\ &= (-0.05-\lambda)((-0.05-\lambda)^2 - 0.05 \times 0.02) \\ &\qquad - 0.05(0.03)(-0.05-\lambda) \\ &= (-0.05-\lambda)(\lambda^2 + .1\lambda + .0025 - .0010 - .0015) \\ &= -(\lambda + 0.05)\lambda(\lambda + .1) \end{aligned}$$

Eigenvalues are $\lambda = 0, -0.05$, and -0.10. For the associated eigenvectors, we find for each λ a solution of $(A - \lambda I)q = 0$. For $\lambda = 0$, we have

$$\begin{pmatrix} -0.05 & 0.03 & 0 \\ 0.05 & -0.05 & 0.05 \\ 0 & 0.02 & -0.05 \end{pmatrix} q = 0.$$

Starting with $q_3 = 1$, the last equation (last row) implies $q_2 = 5/2$ and the first equation implies $q_1 = 3/2$. So, $(3/2, 5/2, 1)$ is an eigenvector corresponding to the eigen value $\lambda = 0$.

For $\lambda = -0.05$, we have

$$\begin{pmatrix} 0 & 0.03 & 0 \\ 0.05 & 0 & 0.05 \\ 0 & 0.02 & 0 \end{pmatrix} q = 0.$$

Starting with $q_3 = 1$, the middle equation implies $q_1 = -1$ and the first and third equations together imply $q_2 = 0$. So, $(-1, 0, 1)$ is an eigenvector corresponding to the eigen value $\lambda = -0.05$.

For $\lambda = -0.10$, we have

$$\begin{pmatrix} 0.05 & 0.03 & 0 \\ 0.05 & 0.05 & 0.05 \\ 0 & 0.02 & 0.05 \end{pmatrix} q = 0.$$

Starting with $q_3 = 1$, the last equation (last row) implies $q_2 = -5/2$ and the first equation implies $q_1 = 3/2$. So, $(3/2, -5/2, 1)$ is an eigenvector corresponding to the eigen value $\lambda = -0.10$.

Since all the eigenvalues are distinct, the homogeneous solution is

$$\pi(t) = K_1 \begin{pmatrix} 3/2 \\ 5/2 \\ 1 \end{pmatrix} + K_2 e^{-.05t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + K_3 e^{-.1t} \begin{pmatrix} 3/2 \\ -5/2 \\ 1 \end{pmatrix}.$$

Since our differential equation is homogeneous, this is also the general solution.

C. Find the solution of the ODE that satisfies the initial condition that we are in a recession at time $t = 0$.

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = K_1 \begin{pmatrix} 3/2 \\ 5/2 \\ 1 \end{pmatrix} + K_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + K_3 \begin{pmatrix} 3/2 \\ -5/2 \\ 1 \end{pmatrix}.$$

From the first and third equations, we can see that $K_2 = 0$. Then from the first or the third equation we can see that $K_1 = -K_3$. Plugging this into the second equation we can see that $K_1 = 1/5$ and $K_3 = -1/5$. So we have the solution

$$\pi(t) = \begin{pmatrix} 0.3 \\ 0.5 \\ 0.2 \end{pmatrix} + e^{-0.1t} \begin{pmatrix} -0.3 \\ 0.5 \\ -0.2 \end{pmatrix}$$

D. We have a possible investment project that requires an initial investment of $\$100,000$. The project pays a cash flow c_t of $\$7,000$/year when the economy is healthy, $\$1,000$/year in a recession, and $\$0$/year in a depression. If the interest rate is 2%, is the expected net present value

$$\int_{t=0}^{\infty} e^{-rt} E[c_t] dt - 100,000$$

of the cash flows positive?

$$\begin{aligned} PV &= \int_{t=0}^{\infty} e^{-rt} E[c_t] dt \\ &= \int_{t=0}^{\infty} e^{-rt} (7000, 1000, 0) \pi(t) dt \\ &= 1000 \int_{t=0}^{\infty} e^{-.02t} (7 \times .3(1 - e^{-.1t}) + 1 \times .5(1 + e^{-.1t}) + 0 \times .2(1 - e^{-.1t})) dt \\ &= 1000 \int_{t=0}^{\infty} (2.6e^{-.02t} - 1.6e^{-.12t}) dt \\ &= 1000 \left(\frac{2.6}{.02} - \frac{1.6}{.12} \right) \\ &= 116,666.67 \end{aligned}$$

So yes, the NPV $(= \$116,666.67 - 100,000 = 16,666.67)$ is positive.