MATHEMATICAL FOUNDATIONS FOR FINANCE

Calculus Review

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Some underyling ideas

Functions: A lot of things in the world can be described by functions. For example, the cost, expected return and variance of a portfolio are all functions of the number of shares we hold of various assets. In this lecture we will work with simple real-valued functions of a single variable: for each real number in its domain, such a function returns a single real number.

Why calculus? Functions can be hard to work with. However, linear (or affine) functions are easy to work with. In calculus, we work with functions by approximating them locally by linear functions, which gives us the tools to do things that would be hard otherwise. A linear function is of the form $f(x) = bx$. An affine function is linear plus a constant, of the form $f(x) = a + bx$. Confusingly, an affine function is sometimes called linear.

Derivatives

The derivative is a local linear approximation to changes in a function. Informally, if $f'(x_0)$ (also written $df(x)/dx$) is the derivative of $f(x_0)$ at x_0, then for x near x_0, there is a good approximation $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$. Doing this formally is more messy:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

where we define a limit by $\lim_{x \to x_0} g(x) = b$ if, for all $\delta > 0$, there exists $\varepsilon > 0$ such that whenever $|x - x_0| < \varepsilon$, $|g(x) - g(x_0)| < \delta$. We say the limit exists if such a b exists as a finite number. This formal definition is not usually needed in practice, but it can be important for doing proofs since it gives the precise sense of the approximation.

graphical interpretations

Integrals

The definite integral is the signed area beneath (above) a curve. The area can be defined as the limit as the grid gets fine of a piecewise linear approximation to the curve (although other definitions are more robust ... not so important for this lecture). We can write the area beneath the curve $y = f(x)$ from $x = a$ to $x = b$ as

$$\int_a^b f(x)dx$$

where areas beneath the x-axis are given negative weight.

The indefinite integral is like an integral with one endpoint free, i.e., $\int f(b)db = \int_a^b(x)dx$ where a is unknown and can be set arbitrarily. If $g(x)$ is any indefinite integral of $f(x)$, then $\int_a^b f(x)dx = [g(x)]_a^b = g(b) - g(a)$. If $g(x)$ is an indefinite integral of $f(x)$, then so is $g(x) + c$ for any constant c, and this is the only form for another indefinite integral of f. Note that the constant cancels out when we compute the definite integral.

graphical interpretations

Fundamental Theorem of Calculus

Theorem: Suppose $f(x)$ is differentiable on its whole domain, which is a connected set. Then $f(x)$ is a definite integral of $f'(x)$.

In general, computing integrals is harder than computing derivatives. If we have some complicated expression and we know how to compute the derivatives of all the pieces, then we can use the chain rule (next slide) to compute the derivative of the complicated expression. There is no general rule for integrals, but we can often use the Fundamental Theorem of Calculus to calculate simple integrals by inspection.

Some formulas for derivatives

function $f(x)$	derivative $f'(x)$
constant C	0
sum $C_1 f(x) + C_2 g(x)$	$C_1 f'(x) + C_2 g'(x)$
power x^n, $n \neq 0$	nx^{n-1}
exponential $e^{ax} = \exp(ax)$	ae^{ax}
logarithm $\log(\lvert x \rvert)$	$1/x$
trig sine $\sin(x)$	$\cos(x)$
trig cosine $\cos(x)$	$-\sin(x)$
$h(g(x))$	$h'(g(x))g'(x)$ chain rule (univariate)
$h(x)g(x)$	$h'(x)g(x) + h(x)g'(x)$ product rule
$h(x)/g(x)$	$h'(x)/g(x) - h(x)g'(x)/(g(x)^2)$ quotient rule

These rules can also be used for computing higher-order derivatives such as $f''(x) = df'(x)/dx = d^2 f(x)/dx^2$. Higher orders can be written $f'''(x)$, $f^{(iv)}(x)$, $f^{(v)}(x)$, etc., using Roman numerals for higher orders.

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*Logarithms in the table are natural logarithms with base e, not logarithms of base 10 often studied in middle school.

In-class exercise: differentiation

- compute the derivative of $f(x) = xe^x$.
- compute the derivative of $\exp(x^2 - 3x)/(x^2 - 3x)$.

l'Hôpital's rule

If $g(y)$ is continuous then

$$\lim_{x \to x_0} g(f(x)) = g(\lim_{x \to x_0} f(x))$$

if the limit on the right-hand side of the equation exists. Similarly,

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)}$$

if both limits on the right-hand side exist and the limit in the denominator is not zero.

Interestingly, the limit for a ratio can be shown to exist in some cases when both limits are 0 or "are" $\pm\infty$. Suppose both limits are zero or $+\infty$, then l'Hôpital's rule tells us that

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

Oftentimes, the limits of the derivatives are finite and nonzero, making this ratio much simpler to compute.

In-class exercise: l'Hôpital's rule

- Compute $\lim_{x\to 0}(\exp(x) - 1)/(\exp(2x) - 1)$
- Compute $\lim_{x\to 0}\sin(x)/x$

Integration

The main idea in doing integration is to put the integrand in the form of the derivative of something, which often is not easy.

Ths simplest integrals are the ones that can be read from the tables. For example, $\int \cos(x)dx = \sin(x) + C$ (where C is the arbitrary integration constant found in an indefinite integral), since we know from the table of derivatives that $d\sin(x)/dx = \cos(x)$. Similarly, $\int(1/x)dx = \log(|x|) + C$ and $\int x^m dx = x^{m+1}/(m+1)$.

Some integrals can be done by putting them in the form $\int f'(g(x))g'(x)dx = f(g(x)) + C$. For example, an integral that comes up in calculations involving means of a normal random variable can be transformed into $\int -xe^{-x^2/2}dx = e^{-x^2/2} + C$ where $g(x) = -x^2/2$ and $f(y) = e^y$.

Integration by parts can also be useful. This says that $\int_a^b U(x)V'(x)dx = [U(x)V(x)]_a^b - \int_a^b V(x)U'(x)dx$. In term structure theory, we often see expressions like $\int_{x=0}^{\infty} x^2 e^x$ that can be evaluated using integration by parts twice (letting $V(x) = e^x$ both times).

Integrals and differential equations

Another way to write the equation $g(x) = \int xe^x dx$ is as the differential equation $g'(x) = xe^x$. A differential equation can have many expressions including the function, the underlying variable, and derivatives of the function, for example,

$$\log(g'''(x)) + \cos(g'(x) + x) = x^2 \exp(x^2 - 3).$$

Our chances of solving such a complicated differential equation analytically are small, but we may be able to find a numerical solution. Typically, such a solution will include some free constants (just as an integral involves an integration constant), and often the free constants can be determined using a boundary condition, such as the initial interest rate or a growth condition on the option value as the stock price increases.

It is useful to know some basic techniques for solving standard differential equations such as the linear equation with constant coefficients

$$af''(t) + bf'(t) + cf(t) = te^{2t} - t^3.$$

There are published tables of exact solutions to differential equations and tables of integrals that can be useful, but of course it is a lot faster if you know how to do these things yourself!

In-class exercise: integration

- Compute $\int f(x)dx$.
- Compute $\int_{x=-1}^{1} x^3 dx$.
- Compute $\int \exp(\sin(x)) \cos(x) dx$.
- Compute $\int_{x=0}^{\infty} xe^x dx$.

Taylor series

quadratic approximation:

$$f(x) \approx f(x_0) + (x - x_0)f'(x_0) + \frac{1}{2}(x - x_0)^2 f''(x_0)$$

K-th order:

$$f(x) \approx \sum_{k=0}^{K} \frac{1}{k!}(x - x_0)^k f^{(k)}(x_0)$$

for example:

$$\exp(x) = \sum_{k=0}^{\infty} \frac{1}{k!}(x - x_0)^k$$