Problem Set 3: Homotheticity and Bellman Equation
FIN 539 Mathematical Finance
P. Dybvig

1. **Homotheticity** Consider the HARA (Hyperbolic Absolute Risk Aversion) felicity (or utility) function $u(c) = (c - \underline{c})^{1-R}/(1-R)$, where $\underline{c}$ is the subsistence consumption (the minimal consumption needed to survive) and $R > 0$, $R \neq 1$, is the relative risk aversion for the increase of consumption above the subsistence level. Then we will study the following optimization problem:

Given w_0,
choose portfolio θ_t, consumption c_t, and wealth w_t to
maximize $E[\int_{t=0}^{\infty} e^{-\rho t} u(c_t)dt]$ (expected utility of lifetime consumption)
subject to:
$dw_t = rw_t dt + \theta_t((\mu - r)dt + \sigma dZ_t) - c_t dt$ (budget constraint)
$(\exists K \in \Re)(\forall t) w_t \geq -K$ (limited borrowing)

Prove that the value function for this problem is of the form $V(w) \equiv (w - \underline{c}/r)^{1-R} v$ for some constant v. Assume that $w_0 > \underline{c}/r$ so there is enough wealth to pay for subsistence consumption with something left over. (Hint: note that wealth of $\underline{c}/r$ can be thought of as being committed to funding the subsistence $\underline{c}$ so "free" or "discretionary" wealth at t is $w_t - \underline{c}$. Define new choice variables: portfolio choice $\hat{\theta} \equiv \theta_t/(w_0 - \underline{c}/r)$ normalized by initial free wealth, consumption $\hat{c}_t \equiv (c_t - \underline{c})/(w_0 - \underline{c}/r)$ in excess of the subsistence consumption, normalized by initial free wealth, and free wealth $\hat{w}_t \equiv (w_t - \underline{c}/r)/(w_0 - \underline{c}/r)$ normalized by initial free wealth.)

2. **Bellman Equation** Consider the optimization problem in Problem 1. (Note: this problem can be solved even if you did not solve Problem 1.)

A. Write down the martingale M_t for this problem.

B. What does M_t represent given the optimal policies for portfolio, consumption, and wealth? What does M_t represent given suboptimal policies? For $t > s$, what is $E[M_s] - E[M_t]$?

C. Derive the Bellman equation for this problem.

D. Solve for optimal c_t and θ_t in terms of derivatives of V, and substitute the optimized values into the Bellman equation.

E. From Problem 1, we can write the value function in the form $V(w) = \frac{k}{1-R}(w - \underline{c}/r)^{1-R}$, where $k = (1-R)v$. Using this formula, solve for the optimal c_t and θ_t in terms of w_t and the parameters.

F. Solve the Bellman equation for k.