Problem Set 4: Multi-asset portfolio problem
FIN 539 Mathematical Finance
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1. **Homotheticity** Consider the log felicity (or utility) function $u(c) = \log(c)$. Then we will study variations of the following multi-asset optimization problem:

Given $w_0$,
choose adapted risky asset proportions $\theta_t$, consumption $c_t$, and wealth $w_t$, to maximize $E[\int_{t=0}^{\infty} e^{-\rho t} u(c_t) dt]$ (expected utility of lifetime consumption)
subject to:
$dw_t = rw_t dt + w_t \theta_t'((\mu - r\mathbf{1})dt + \Gamma dZ_t) - c_t dt$ (budget constraint)
$w_t \geq 0$ (no borrowing)

The choice variables are three processes: the vector of risky asset proportions $\theta_t \in \Re^N$, real-valued consumption $c_t$, and real-valued wealth $w_t$. The constant $\rho$ is the pure rate of time discount, the constant $r$ is the instantaneous riskfree rate of interest, $\mu \in \Re^N$ is the constant vector of mean risky asset returns, $\Gamma$ is the constant $N \times k$ matrix of loadings of the returns on the different risks, and $\mathbf{1}$ is the $N$-vector of 1's. Assume the local covariance $\Gamma\Gamma'$ of returns is positive definite, and that there is at least one asset $n$ with $\mu_n > r$.

A. Show that the form of the value function for this problem is $V(w) = v + \log(w)/\rho$ for some constant $v$.

Let $\hat{\theta}_t \equiv \theta_t$ (no need to change since it is already normalized by wealth), $\hat{c}_t \equiv c_t/w_0$, and $\hat{w}_t \equiv w_t/w_0$. Then we can write the problem as

Given $w_0$,
choose adapted $\hat{\theta}_t$, $\hat{c}_t$, and $\hat{w}_t$ to
maximize $E[\int_{t=0}^{\infty} e^{-\rho t} \log(w_0 \hat{c}_t) dt]$
subject to:
$d\hat{w}_t = r\hat{w}_t dt + \hat{w}_t \hat{\theta}_t'((\mu - r\mathbf{1})dt + \Gamma dZ_t) - \hat{c}_t dt$
$\hat{w}_t \geq 0$ (no borrowing)

The objective function can be written as

$$
\begin{aligned}
E[\int_{t=0}^{\infty} e^{-\rho t}\log(w_0\hat{c}_t)dt] &= \int_{t=0}^{\infty} e^{-\rho t}\log(w_0)dt + E[\int_{t=0}^{\infty} e^{-\rho t}\log(\hat{c}_t)dt] \\
&= \log(w_0)/\rho + E[\int_{t=0}^{\infty} e^{-\rho t}\log(\hat{c}_t)dt].
\end{aligned}
$$

Writing the problem this way, the constraints do not depend on $w_0$ and the objective function depends on $w_0$ only through the additive constant $\log(w_0)/\rho$. Therefore, the optimal choices of $\hat{\theta}_t$, $\hat{c}_t$, and $\hat{w}_t$ do not depend on $w$ and we have that $V(w) = v + \log(w)/\rho$, where $v$ is the optimized value of the second term, which is $V(1)$.

B. Does the result in part A hold (perhaps for a different constant $v$) if we add the constraint

$$(\forall i, t)(0 \leq \theta_{it} \leq K_i)$$

where each $K_i > 0$ is a given constant? Explain why or why not. (If not, it suffices to show where the usual argument breaks down.)

Yes, it does, since $\hat{\theta}_t = \theta_t$ and the new constraints in terms of the transformed variables do not depend on $w$.

C. Does the result in part A hold (perhaps for a different constant $v$) if we add the constraint

$$(\forall t)(0 \leq \theta_t \leq w_t K)$$

where $K > 0$ is a given constant. Explain why or why not. (If not, it suffices to show that the usual argument breaks down.)

No, because the constraint in the transformed variables becomes

$$(\forall t)(0 \leq \hat{\theta}_t \leq w\hat{w}_t K),$$

which does depend on $w$. In particular, this is very binding when $w$ is very small, but not very binding when $w$ is large. Since there is some asset with $\mu_i > r$, holding a small positive amount of that asset would dominate just holding the riskless asset, so the constraint is strictly binding for $w$ sufficiently small.

2. **Bellman Equation** Consider the optimization problem in Problem 1, without either constraint described in Part B or Part C. (Note: this problem can be solved even if you did not solve Problem 1.)

A. Write down the process $M_t$ for this problem.

$$M_t \equiv \int_{s=0}^{t} e^{-\rho s}\log(c_s)ds + e^{-\rho t}V(w_t)$$

B. What does $M_t$ represent given the optimal policies for portfolio, consumption, and wealth? What does $M_t$ represent given suboptimal policies? For $t < s$, what is $E[M_t] - E[M_s]$?

Given the optimal policy, $M_t$ is the conditional expectation at $t$ of the "realized utility" (the quantity in the expectation of the objective function) for the optimal strategy. Given a suboptimal strategy, it is the conditional expectation at $t$ of following the suboptimal strategy until $t$ and then the optimal strategy from then on. $E[M_t] - E[M_s]$ is the decrease in the objective function due to mistakes made between time $t$ and time $s$.

C. Derive the Bellman equation for this problem.

Use Itô's lemma and the budget constraint to compute $dM_t$:

$$M_t = \int_{s=0}^{t} e^{-\rho s}\log(c_s)ds + e^{-\rho t}V(w_t),$$

we can compute

$$E\left[\frac{dM}{e^{-\rho t}dt}\right] \quad = \quad \log(c) - \rho V + (rw + w\theta'(\mu - r\mathbf{1}) - c)V_w$$

$$+\frac{1}{2}\mathrm{tr}(w\theta'\Gamma\Gamma'\theta w V_{ww}),$$

where $\theta'\Gamma\Gamma'\theta$, $w$, and $V_{ww}$ are scalars, so the Bellman equation is

$$\max_{c,\theta}\left(\log(c) - \rho V + ((r + \theta'(\mu - r\mathbf{1}))w - c)V_w + \frac{w^2 V_{ww}}{2}\theta'\Gamma\Gamma'\theta\right) = 0$$

D. Solve for optimal $c_t$ and $\theta_t$ in terms of derivatives of $V$.

The terms involving $\theta$ are

$$\theta'(\mu - r\mathbf{1})wV_w + \frac{w^2 V_{ww}}{2}\theta'\Gamma\Gamma'\theta,$$

where $\Gamma\Gamma'$ is the local covariance matrix of security returns. The first-order condition for optimal $\theta$ is

$$(\mu - r\mathbf{1})wV_w + w^2\Gamma\Gamma'\theta V_{ww} = 0$$

As before, $u'(c) = V_w$ so $c = I(V_w)$, but now the optimal portfolio is

$$\theta^* = \frac{1}{-wV_{ww}/V_w}(\Gamma\Gamma')^{-1}(\mu - r\mathbf{1})$$

The optimization is locally a mean-variance problem. Note the coefficient of relative risk aversion of the value function in the denominator.

E. From Problem 1, we can write the value function in the form $V(w) = v + \log(w)/\rho$. Using this formula, solve for the optimal $c_t$ and $\theta_t$ in terms of $w_t$ and the parameters.

For some $v$,

$$V(w) = v + \log(w)/\rho$$

$$V_w(w) = \frac{1}{\rho w}$$

$$V_{ww}(w) = -\frac{1}{\rho w^2}$$

Now, $u(c) = \log(c)$, $u'(c) = 1/c$, and $I(z) = 1/z$. Therefore,

$$c_t{}^* = \frac{1}{V_w(w_t)} = \rho w$$

$$\begin{aligned} \theta_t{}^* &= \frac{1}{-wV_{ww}/V_w}(\Gamma\Gamma')^{-1}(\mu - r\mathbf{1}) \\ &= (\Gamma\Gamma')^{-1}(\mu - r\mathbf{1}). \end{aligned}$$

F. Substitute the optimal portfolio and consumption policies into the Bellman equation, and solve the optimized Bellman equation for $v$.

$$\begin{aligned} 0 &= \log(\rho w) - \rho(v + \log(w)/\rho) + ((r + (\mu - r\mathbf{1})'(\Gamma\Gamma')^{-1}(\mu - r\mathbf{1}))w \\ &\quad -\rho w)\frac{1}{\rho w} - \frac{w^2}{2\rho w^2}(\mu - r\mathbf{1})'(\Gamma\Gamma')^{-1}\Gamma\Gamma'(\Gamma\Gamma')^{-1}(\mu - r\mathbf{1}) \\ &= \log(\rho) + \log(w) - \rho v - \log(w) + \frac{r-\rho}{\rho} \\ &\quad + \frac{1}{2\rho}(\mu - r\mathbf{1})'(\Gamma\Gamma')^{-1}(\mu - r\mathbf{1}). \end{aligned}$$

Therefore,

$$v = \frac{1}{\rho}\left(\log(\rho) + \frac{r-\rho}{\rho} + \frac{1}{2\rho}(\mu - r\mathbf{1})'(\Gamma\Gamma')^{-1}(\mu - r\mathbf{1})\right)$$

Recall that this is the value function $V(1)$ when $w = 1$. The first term in parentheses is what we get if wealth never changes and we consume $\rho$ forever. The second term is the value of consumption changing over time because the interest $rw$ funds more or less than the consumption $\rho w$. The third term in parentheses is the value of investing in the market, which is larger the further $\mu$ is from $r\mathbf{1}$ and smaller the larger the variances are.