FIN 550J Exam, October 20, 2011

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This is a closed-book examination. You may not use texts, notes, a crib sheet, calculator, cell phone, listening device, or any other electronics. Answer all questions as directed. Make sure each answer is clearly indicated.

0. PLEDGE

The work on this exam is my own alone, and I have conformed with the rules of the exam and the code of the conduct of the Olin School.

Signed name ____

Printed name (write clearly) ____

1. LINEAR EQUATIONS (10 points) Consider the system of equations:

$$x_1 = 17 - 2x_2 - 3x_3 + x_4$$
$$x_2 = 8 - 2x_1 - x_3 - x_2$$
$$x_3 - 5 = x_1 - x_2$$
$$x_4 + 2 = x_3 + 3$$

Write these equations in the form $Ax = b$. What are A and b? DO NOT SOLVE for x.

2. PROBABILITIES (25 points) A position long one digital option and short another can has a payoff that is 1 with probability 1/5, 0 with probability 3/5, and -1 with probability 1/5. Compute the mean, variance, standard deviation, skewness, and kurtosis of the payoff.

3. EIGENVALUES AND EIGENVECTORS (30 points) Let

$$M = \begin{pmatrix} \frac{17}{2} & 0 & \frac{15}{2} \\ 0 & 2 & 0 \\ \frac{15}{2} & 0 & \frac{17}{2} \end{pmatrix}$$

a. Confirm that

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \; x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \text{ and } x_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

are eigenvectors of M, and compute the corresponding eigenvalues.

b. Use the eigenvalues to compute the determinant of M.

c. Compute the definiteness of M. In other words, is M positive definite, positive semi-definite, negative definite, negative semi-definite, or indefinite?

d. Compute

$$M^6 \begin{pmatrix} 10 \\ 1 \\ -10 \end{pmatrix}$$

4. OPTIMIZATION (35 points) Solve the following maximization problem:

Choose x_1 and x_2 to
maximize $2\log(x_1) + \log(x_2)$
subject to
$x_1 + x_2 \leq 30$
and
$15 \leq x_1$.

Note: the second-order conditions are satisfied because the Hessian of the objective function is negative definite and the constraint set is convex. You do not need to prove this.[1]

[1]Implicitly, $x_1 > 0$ and $x_2 > 0$ or else the objective function would not be defined. However, these strict inequality constraints do not have to be included because they cannot be binding at the optimum.

HELPFUL FORMULAS

$Ax = \lambda x$
$\det(A - \lambda I) = 0$
$(A - \lambda I)x = 0$

Choose $x \in \Re^N$ to
maximize $f(x)$
subject to $(\forall i \in \mathcal{E}) g_i(x) = 0$, and
$(\forall i \in \mathcal{I}) g_i(x) \leq 0$.

$\nabla f(x^*) = \sum_{i \in \mathcal{E} \bigcup \mathcal{I}} \lambda_i \nabla g_i(x^*)$
$(\forall i \in \mathcal{I}) \lambda_i \geq 0$
$\lambda_i g_i(x^*) = 0$