FIN 550J Exam, October 20, 2011

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This is a closed-book examination. You may not use texts, notes, a crib sheet, calculator, cell phone, listening device, or any other electronics. Answer all questions as directed. Make sure each answer is clearly indicated.

0. PLEDGE

The work on this exam is my own alone, and I have conformed with the rules of the exam and the code of the conduct of the Olin School.

Signed name ____

Printed name (write clearly) ____

1. LINEAR EQUATIONS (10 points) Consider the system of equations:

$$x_1 = 17 - 2x_2 - 3x_3 + x_4$$
$$x_2 = 8 - 2x_1 - x_3 - x_2$$
$$x_3 - 5 = x_1 - x_2$$
$$x_4 + 2 = x_3 + 3$$

Write these equations in the form $Ax = b$. What are A and b? DO NOT SOLVE for x.

$$x_1 + 2x_2 + 3x_3 - x_4 = 17$$
$$2x_1 + 2x_2 + x_3 + 0x_4 = 8$$
$$-x_1 + x_2 + x_3 + 0x_4 = 5$$

$$0x_1 + 0x_2 - x_3 + x_4 = 1$$

$$A = \begin{pmatrix} 1 & 2 & 3 & -1 \\ 2 & 2 & 1 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, b = \begin{pmatrix} 17 \\ 8 \\ 5 \\ 1 \end{pmatrix}$$

2. PROBABILITIES (25 points) A position long one digital option and short another can has a payoff that is 1 with probability 1/5, 0 with probability 3/5, and -1 with probability 1/5. Compute the mean, variance, standard deviation, skewness, and kurtosis of the payoff.

$$\text{mean} = \frac{1}{5} \times 1 + \frac{3}{5} \times 0 + \frac{1}{5} \times (-1) = 0$$

$$\text{variance} = \frac{1}{5} \times (1-0)^2 + \frac{3}{5} \times (0-0)^2 + \frac{1}{5} \times (-1-0)^2 = \frac{2}{5}$$

$$\text{standard deviation} = \sqrt{\text{variance}} = \sqrt{\frac{2}{5}}$$

$$\text{skewness} = \frac{\frac{1}{5} \times (1-0)^3 + \frac{3}{5} \times (0-0)^3 + \frac{1}{5} \times (-1-0)^3}{\text{standard deviation}^3} = 0$$

$$\text{kurtosis} = \frac{\frac{1}{5} \times (1-0)^4 + \frac{3}{5} \times (0-0)^4 + \frac{1}{5} \times (-1-0)^4}{\text{standard deviation}^4} = \frac{2/5}{(2/5)^2} = \frac{5}{2}$$

3. EIGENVALUES AND EIGENVECTORS (30 points) Let

$$M = \begin{pmatrix} \frac{17}{2} & 0 & \frac{15}{2} \\ 0 & 2 & 0 \\ \frac{15}{2} & 0 & \frac{17}{2} \end{pmatrix}$$

a. Confirm that

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \; x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \text{ and } x_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

are eigenvectors of M, and compute the corresponding eigenvalues.

$$\begin{pmatrix} \frac{17}{2} & 0 & \frac{15}{2} \\ 0 & 2 & 0 \\ \frac{15}{2} & 0 & \frac{17}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{17}{2} \times 1 + 0 \times 0 + \frac{15}{2} \times 1 \\ 0 \times 1 + 2 \times 0 + 0 \times 1 \\ \frac{15}{2} \times 1 + 0 \times 0 + \frac{17}{2} \times 1 \end{pmatrix} = \begin{pmatrix} 16 \\ 0 \\ 16 \end{pmatrix} = 16 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

confirming x_1 is an eigenvector with eigenvalue $\lambda_1 = 16$

$$\begin{pmatrix} \frac{17}{2} & 0 & \frac{15}{2} \\ 0 & 2 & 0 \\ \frac{15}{2} & 0 & \frac{17}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{17}{2} \times 0 + 0 \times 1 + \frac{15}{2} \times 0 \\ 0 \times 0 + 2 \times 1 + 0 \times 0 \\ \frac{15}{2} \times 0 + 0 \times 1 + \frac{17}{2} \times 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

confirming x_2 is an eigenvector with eigenvalue $\lambda_2 = 2$

$$\begin{pmatrix} \frac{17}{2} & 0 & \frac{15}{2} \\ 0 & 2 & 0 \\ \frac{15}{2} & 0 & \frac{17}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{17}{2} \times 1 + 0 \times 0 + \frac{15}{2} \times (-1) \\ 0 \times 1 + 2 \times 0 + 0 \times (-1) \\ \frac{15}{2} \times 1 + 0 \times 0 + \frac{17}{2} \times (-1) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

confirming x_3 is an eigenvector with eigenvalue $\lambda_3 = 1$

b. Use the eigenvalues to compute the determinant of M.

The determinant is the product of the eigenvalues, $16 \times 2 \times 1 = 32$.

c. Compute the definiteness of M. In other words, is M positive definite, positive semi-definite, negative definite, negative semi-definite, or indefinite?

All eigenvalues are positive, so M is positive definite.

d. Compute

$$M^6 \begin{pmatrix} 10 \\ 1 \\ -10 \end{pmatrix}$$

$$M^6 \begin{pmatrix} 10 \\ 1 \\ -10 \end{pmatrix} = M^6 \left(x_2 + 10x_3\right) = M^6 x_2 + 10 M^6 x_3 = 2^6 x_2 + 1^6 x_3$$

$$= \begin{pmatrix} 10 \\ 64 \\ -10 \end{pmatrix}$$

4. OPTIMIZATION (35 points) Solve the following maximization problem:

Choose x_1 and x_2 to
maximize $2\log(x_1) + \log(x_2)$
subject to
  $x_1 + x_2 \leq 30$
and
  $15 \leq x_1$.

Note: the second-order conditions are satisfied because the Hessian of the objective function is negative definite and the constraint set is convex. You do not need to prove this.[1]

$$\nabla f(x) = \left(\frac{2}{x_1}, \frac{1}{x_2}\right)^T$$

[1]Implicitly, $x_1 > 0$ and $x_2 > 0$ or else the objective function would not be defined. However, these strict inequality constraints do not have to be included because they cannot be binding at the optimum.

$$g_1(x) = x_1 + x_2 - 30 \Rightarrow \nabla g_1(x) = (1,1)^T$$
$$g_2(x) = 15 - x_1 \Rightarrow \nabla g_2(x) = (-1,0)^T$$

The Kuhn-Tucker conditions are

$$\nabla f(x) = \lambda_1 \nabla g_1(x) + \lambda_2 \nabla g_2(x)$$
$$\lambda_1 g_1(x) = 0$$
$$\lambda_2 g_2(x) = 0$$
$$\lambda_1 \geq 0$$
$$\lambda_2 \geq 0$$

The question is which constraints are binding, which of the multipliers is zero. If $\lambda_1 = \lambda_2 = 0$, neither constraint is binding and we would have to have $\nabla f(x) = 0$, i.e. $2/x_1 = 0$ and $1/x_2 = 0$, which cannot be.

If $\lambda_1 \neq 0$ and $\lambda_2 = 0$, then $\nabla f(x) = \lambda_1(1,1)$, i.e., $2/x_1 = \lambda_1$ and $1/x_1 = \lambda_1$. Therefore, $x_1 = 2/\lambda_1$ and $x_2 = 1/\lambda_1$. Since the first constraint is binding (by complementary slackness), $x_1 + x_2 = 30$, so $1/\lambda_1 = 10$. This implies $x_1 = 20$ and $x_2 = 10$. Since these choices satisfies the first constraint with equality and the second with inequality, and $\lambda_1 > 0$ and $\lambda_2 = 0$, this satisfies the constraints and all the Kuhn-Tucker conditions and we have found the optimum.