Problem Set 2: Probability using Calculus
FIN 500J Mathematical Foundations for Finance
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Prepare a hard copy of your answers to submit in class. It is okay to work with others on the problems, but do the write-up and any computer work yourself. Show your work.

1. Suppose the interest rate $r$ a year from now has the density function

$$f(r) = \begin{cases} \frac{k}{r^4} & \text{for } r > 0.01 \\ 0 & \text{for } r < 0.01 \end{cases}$$

where $k$ is a constant.

a. Compute the constant $k$.

$$k = .000003$$

b. Compute $E[r]$.

$$r = \int_{r=.01}^{\infty} r f(r) dr = .015$$

c. Compute $\text{var}(r)$ and $\text{std}(r)$.

$$\text{var}(r) = E[r^2] - (E[r])^2 = .000075$$

$$\text{std}(r) = \sqrt{\text{var}(r)} \approx .0866$$

2. Assume the stock price $S$ three months from now has an exponential distribution with scale parameter $\theta > 0$ as described in the slides, i.e. the density of $S$ is

$$f(S) = \begin{cases} \frac{1}{\theta} e^{-S/\theta} & \text{for } S > 0 \\ 0 & \text{for } S < 0 \end{cases}$$

and the cumulative distribution function of $S$ is

$$F(S) = \begin{cases} 1 - e^{-S/\theta} & \text{for } S > 0 \\ 0 & \text{for } S < 0 \end{cases}$$

Consider a call option on this stock maturing three months from now with a strike price $X > 0$. The payoff of the call option is

$$C = \max(S - X, 0).$$

A. What is the cumulative distribution function of the option payoff?

$$\begin{aligned}
G(C) &= \text{prob}(\max(S-X,0) < C) \\
&= \begin{cases} \text{prob}(X < X + C) & \text{for } C \geq 0 \\ 0 & \text{for } C < 0 \end{cases} \\
&= \begin{cases} F(X + C) & \text{for } C \geq 0 \\ 0 & \text{for } C < 0 \end{cases} \\
&= \begin{cases} 1 - e^{-(X+C)/\theta} & \text{for } C \geq 0 \\ 0 & \text{for } C < 0 \end{cases}
\end{aligned}$$

We can see that $C$ is never negative, $C = 0$ with probability $1 - e^{-X/\theta}$, and $C > 0$ with positive density $g(C) = G'(C) = (1/\theta)e^{-(X+C)/\theta}$.

B. What is the expected option payoff?

$$\begin{aligned}
E[C] &= (1 - e^{-X/\theta})0 + \int_{C=0}^{\infty} \frac{1}{\theta} e^{-(X+C)/\theta} C dC \\
&= 0 + e^{-X/\theta} \int_{C=0}^{\infty} \frac{1}{\theta} e^{-C/\theta} C dC. \\
&\text{now integrate by parts: } U = C \text{ and } V = -e^{-C/\theta} \\
&= e^{-X/\theta} \left( \left[ Ce^{-C/\theta} \right]_0^{\infty} + \int_{C=0}^{\infty} e^{-C/\theta} dC \right) \\
&= e^{-X/\theta} \left( 0 + \left[ -\theta e^{-C/\theta} \right]_0^{\infty} \right) \\
&= \theta e^{-X/\theta}
\end{aligned}$$

3. Suppose the random $x$ and $y$ have the joint density function

$$p(x, y) = \begin{cases} 2 - x - y & \text{if } 0 < x < 1 \text{ and } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

a. Compute $E[x]$ and $E[y]$.

$$E[x] = E[y] = \frac{5}{12} \approx .4167$$

b. Compute $\text{cov}(x, y)$.

$$\text{cov}(x, y) = E[xy] - E[x]E[y] = -\frac{1}{144} \approx -.0069$$

c. Are $x$ and $y$ independent?