Supplemental notes: Kuhn-Tucker first-order conditions
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Minimization problem (like in the slides):

Choose $x \in \Re^N$ to
minimize $f(x)$
subject to $(\forall i \in \mathcal{E}) g_i(x) = 0$, and
$(\forall i \in \mathcal{I}) g_i(x) \geq 0$.

$x = (x_1, ..., x_N)$ is a vector of *choice variables.*
$f(x)$ is the scalar-valued *objective function.*
$g_i(x) = 0$, $i \in \mathcal{E}$ are *equality constraints.*
$g_i(x) \geq 0$, $i \in \mathcal{I}$ are *inequality constraints.*
$\mathcal{E} \bigcap \mathcal{I} = \emptyset$

Kuhn-Tucker conditions:

$\nabla f(x^*) = \sum_{i \in \mathcal{E} \bigcup \mathcal{I}} \lambda_i \nabla g_i(x^*)$
$(\forall i \in \mathcal{I}) \lambda_i \geq 0$
$\lambda_i g_i(x^*) = 0$

The feasible solution x^* is called *regular* if the set $\{\nabla g_i(x^*) | g_i(x^*) = 0\}$ is a linearly independent set. In particular, an interior solution is always regular.

If x^* is regular and f and the g_is are differentiable, the Kuhn-Tucker conditions are necessary for feasible x^* to be optimal. If the optimization problem is convex, then the Kuhn-Tucker conditions are sufficient for an optimum.

This minimization problem is convex if the objective is convex ($f''(x)$ positive semidefinite everywhere) and the constraint set is convex. The constraint set is convex if $g_i(x)$ is affine for all $i \in \mathcal{E}$ and $g_i(x)$ is concave ($g_i(x)$ is negative semidefinite) for all $i \in \mathcal{I}$.

Maximization problem:

Choose $x \in \Re^N$ to
maximize $f(x)$
subject to $(\forall i \in \mathcal{E}) g_i(x) = 0$, and
$(\forall i \in \mathcal{I}) g_i(x) \leq 0$.

$x = (x_1, ..., x_N)$ is a vector of *choice variables*.
$f(x)$ is the scalar-valued *objective function*.
$g_i(x) = 0$, $i \in \mathcal{E}$ are *equality constraints*.
$g_i(x) \leq 0$, $i \in \mathcal{I}$ are *inequality constraints*.
$\mathcal{E} \bigcap \mathcal{I} = \emptyset$

Kuhn-Tucker conditions:

$\nabla f(x^*) = \sum_{i \in \mathcal{E} \bigcup \mathcal{I}} \lambda_i \nabla g_i(x^*)$
$(\forall i \in \mathcal{I}) \lambda_i \geq 0$
$\lambda_i g_i(x^*) = 0$

(same theorems as on the previous page)

This maximization problem is convex if the objective is concave ($f''(x)$ negative semidefinite everywhere) and the constraint set is convex. The constraint set is convex if $g_i(x)$ is affine for all $i \in \mathcal{E}$ and $g_i(x)$ is convex ($g_i(x)$ is positive semidefinite) for all $i \in \mathcal{I}$.