MATHEMATICAL FOUNDATIONS FOR FINANCE

Introduction to Probability and Statistics: Part 2

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Continuous random variables

Discrete random variables take on isolated values, while continuous random variables can take on all values in some interval or intervals. Which to use is a modelling choice often based on convenience; taking on a thousand different values may not be qualitatively different from taking on any value in a continuum.

Work with a continuous random variable x, we often work with the probability density function $f(x)$ or the cumulative distribution function $F(x)$. The probability density function tells the density of the probability measure with respect to x; we can say informally that $f(x_0)\Delta x$ is the approximate probability of x being in an interval of length Δx at x_0.

The cumulative distribution function $F(x_0)$ at x_0 is the probability x will take on a value less than x_0, and it is related to the density function $f(x)$ by $F(x_0) = \int_{x=-\infty}^{x_0} f(x)dx$. We can express the expectation of a function h of x in terms of F or f as

$$E[h(x)] = \int_{x=-\infty}^{\infty} h(x)f(x)dx = \int_{x=-\infty}^{\infty} h(x)dF(x).$$

Properties of density and distributions functions

- If the probability density function $f(x)$ exists everywhere, then
  - $(\forall x) f(x) \geq 0$
  - $\int_{x=-\infty}^{\infty} f(x) = 1$
  - $(\forall x_0) F(x_0) = \int_{x=-\infty}^{x_0} f(x)$
  - $f(x)$ is not defined for discrete random variables.
- The probability distribution function has the the properties:
  - $(\forall x_0 \leq x_1) F(x_1) \geq F(x_0)$
  - $\lim_{x \to -\infty} F(x) = 0$
  - $\lim_{x \to \infty} F(x) = 1$
  - $F(x)$ is defined for all random variables, discrete or continuous.

Using the density to compute moments

We know that $E[h(x)] = \int_{x=-\infty}^{\infty} h(x)f(x)dx$. In particular, this allows us to compute moments like the mean ($h(x) = x$) and the variance ($h(x) = (x-\mu_x)^2$), and other statistics like the skewness and kurtosis that can be written as functions of moments. Specifically,

$$\mu_x = E[x] = \int_{x=-\infty}^{\infty} xf(x)dx.$$

$$\text{var}(x) = E[x^2] - {\mu_x}^2 = \left(\int_{x=-\infty}^{\infty} x^2 f(x)dx\right) - {\mu_x}^2$$

$$\text{skew}(x) = \frac{E[(x-\mu_x)^3]}{{\sigma_x}^3} = \frac{\int_{x=-\infty}^{\infty}(x-\mu_x)^3 f(x)dx}{\text{var}(x)^{3/2}}$$

$$\text{kurt}(x) = \frac{E[(x-\mu_x)^4]}{{\sigma_x}^4} = \frac{\int_{x=-\infty}^{\infty}(x-\mu_x)^4 f(x)dx}{\text{var}(x)^2}$$

In-class exercise: moments from a density function

Let $\lambda > 0$ be a given constant. Suppose x has the exponential distribution of the form $f(x) = k \exp(-\lambda x)$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$.

- Compute the constant k.
- Compute $E[x]$.
- Compute $\mathrm{var}(x)$.
- Compute the cumulative distribution function $F(x)$.

Normal distribution

The normal distribution (traditional bell curve) with parameters μ and σ has the density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{(x-\mu)^2}{2\sigma^2}).$$

A random variable x with this density has mean μ, standard deviation σ, and variance σ^2. Sometimes this density $f(x)$ is written as $n(x)$, and the corresponding cumulative distribution function $F(x)$ is written as $N(x)$. We can write $N(x_0) = \int_{x=-\infty}^{x_0} n(x)dx$ (as always), but we do not know a closed form expression for N.

The normal distribution plays an important part in statistical theory, because if we have a large number T of independent and identically distributed (i.i.d.) draws from a random variable y with mean m and standard deviation s, then $\left(\Sigma_{t=1}^{T}(y_t - m)\right)/(s\sqrt{T})$ is approximately distributed normally with mean $\mu = 0$ and standard deviation $\sigma = 1$. This fact is the starting point of many statistical tests.

Normal distribution: moment generating function, skewness and kurtosis

The moment generating function of a random variable x is given by

$$M(t) = E[e^{tX}].$$

$M(t)$ is called the moment generating function because the n^{th} moment of x around the origin is equal to the n^{th} derivative of $M(t)$ evaluated at 0. The moment generating function is also useful in finance because it allows us to calculate the expected utility of an exponential utility function.[1]

For a normal random variable x with mean parameter μ and standard deviation parameter σ, the moment generating function is $e^{\mu t+\sigma^2 t^2/2}$. This can be used to verify the mean μ, variance σ^2, skewness 0, and kurtosis 3 of x.

[1] Arguably, the characteristic function, $E[e^{itx}]$ (sometimes defined as a constant times this), where $i^2 = -1$, is more useful in general because it exists for all distributions. However, the moment generating function is suitable for our purposes.

In-class exercise: normal moment generating function

Let x be a random variable with moment generating function $M(t) = e^{\mu t + \sigma^2 t^2/2}$. Show that

- x has mean μ
- x has variance σ

Consider the utility funcion $u(w) = -e^{-Aw}$ (exponential utility with absolute risk aversion A). Derive $E[u(x)]$.

Extra for experts: show that the skewness of x is 0.

Joint distributions and joint normal density

If we have more than one continuous random variable x_1, ..., x_n, we can use the joint density $f(x_1, ..., x_n)$ for all the random variables to write expectations of functions of all of them:

$$E[h(x_1, ..., x_n)] = \int_{x_1=-\infty}^{\infty} \int_{x_1=-\infty}^{\infty} ... \int_{x_1=-\infty}^{\infty} h(x_1, ..., x_n) dx_1 dx_2 ... dx_n.$$

One example of this is the bivariate joint normal distribution, which has density

$$\begin{aligned} f(x, y) \; = \; & \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left(\frac{-1}{2(1-\rho^2)}\left(\frac{(x-\mu_x)^2}{{\sigma_x}^2}\right.\right. \\ & \left.\left. - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \frac{(y-\mu_y)^2}{{\sigma_y}^2}\right)\right) \end{aligned}$$

In this density, the parameters μ_x, μ_y, σ_x, and σ_y are what they seem to be. The parameter ρ is the correlation coefficient, which equals

$$\rho = \frac{\text{cov}(x, y)}{\sigma_x\sigma_y}$$

and takes on values between -1 and 1 (for all random variables, not just jointly normal variables).

Independence

Two random variables are independent if knowing about one random variable gives us no information about the other. This would be true (or at least a very good approximation) for repeated rolls of dice or flips of a coin, and it is a good approximation for returns in some security markets.

In terms of the distribution functions, x and y are independent if the joint probability density is multiplicatively separable:

$$p(x, y) = f(x)g(y)$$

In this case, $f(x)$ is the density function of x and $f(y)$ is the density function of y.

In-class exercise: independence and joint normality

Prove that x and y are independent if they are joint normally distributed with zero correlation ($\rho = 0$).