MATHEMATICAL FOUNDATIONS FOR FINANCE

Quadratic Taylor Series Approximation

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Background

It is often useful to approximate a function locally using derivatives. For example, suppose you are providing analytical screens for traders and you are using a proprietary model to compute a matrix of option prices for many stocks and bonds. This may take a lot of computing time to do this for the current stock prices and vols, but traders need to be able to know how the values change as conditions change during the day. Having a local approximation to the impact of vol and price of the underlying on option prices can be a quick and reasonably accurate subsitution to re-running the whole model.

Probably, something should be in place to help traders in the event of a big move, but that is not the focus today. Methods such as a spline fit, Padé approximants, Fourier analysis, or wavelets can be used to take finitely many valuations on a grid and use them to approximate the whole function.

A quadratic approximation is also an important part of many modern optimization methods, for example Newton-Raphson.

Univariate Case

$$f(x) \approx f(x_0) + (x - x_0)f'(x_0) + \frac{1}{2}(x - x_0)^2 f''(x_0)$$

K-th order:

$$f(x) \approx \sum_{k=0}^{K} \frac{1}{k!}(x - x_0)^k f^{(k)}(x_0)$$

Multivariate case

$$f(x) \approx f(x_0) + (x - x_0)^T f'(x_0) + \frac{1}{2}(x - x_0)^T f''(x)(x - x_0)$$

- x is an $n \times 1$-vector.
- $f : \Re^n \longrightarrow \Re$ is a real-valued function.
- $f'(x) = \nabla f : \Re^n \longrightarrow \Re$ is the gradient.
- $f''(x) : \Re^n \longrightarrow \Re^{n \times n}$ is the Jacobian.

K-th order:

requires n-forms (like multidimensional array)
we don't have notation for that.