Problem 4. For each of the following functions, find the critical points and classify these as local max, local min, saddle point or 'can't tell':

$$(1) xy^2 + x^3y - xy, \quad (2) x^2 + 6xy + y^2 - 3yz + 4z^2 + 6x + 17y - 2z.$$

Solution : (1) First order conditions are:

$$(i) y^2 + 3x^2y - y = 0, \quad (ii) 2xy + x^3 - x = 0.$$

From (i), we get: $y = 1 - 3x^2$ or $y = 0$. From (ii), we get: $x = 0$ or $x^2 = 1 - 2y$. It is easy to see that there are six critical points:

$$(x^*, y^*) = (0,0), (0,1), (1,0), (-1,0), (\frac{\sqrt{5}}{5}, \frac{2}{5}), or (-\frac{\sqrt{5}}{5}, \frac{2}{5}).$$

The Hessian matrix of $xy^2 + x^3y - xy$ is:

$$H = \begin{pmatrix} 6xy & 2y + 3x^2 - 1 \\ 2y + 3x^2 - 1 & 2x \end{pmatrix}$$

at $(0,0), (0,1), (1,0)$ and $(-1,0)$,

$$H = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix}, \text{ } and \begin{pmatrix} 0 & 2 \\ 2 & -2 \end{pmatrix}$$

respectively, it is straightforward to check that these matrix are indefinite, so $(0,0), (0,1)$, $(1,0)$ and $(-1,0)$ are saddle points. At $(\frac{\sqrt{5}}{5}, \frac{2}{5})$,

$$H = \begin{pmatrix} \frac{12\sqrt{5}}{25} & \frac{2}{5} \\ \frac{2}{5} & \frac{2\sqrt{5}}{5} \end{pmatrix}$$

is positive definite, so $(\frac{\sqrt{5}}{5}, \frac{2}{5})$ is a local minimum. At $(-\frac{\sqrt{5}}{5}, \frac{2}{5})$,

$$H = \begin{pmatrix} -\frac{12\sqrt{5}}{25} & \frac{2}{5} \\ \frac{2}{5} & -\frac{2\sqrt{5}}{5} \end{pmatrix}$$

is negative definite, so $(-\frac{\sqrt{5}}{5}, \frac{2}{5})$ is a local maximum.

(2)First order conditions are:

$$(i) 2x + 6y + 6 = 0, \quad (ii) 6x + 2y - 3z + 17 = 0, \quad (iii) - 3y + 8z - 2 = 0.$$

From (i), we get: $(iv) x = -3y - 3$. From (iii), we get: $(v) z = \frac{1}{8}(3y + 2)$. Plugging (iv) and (v) into (ii), we can solve for $y^* = -\frac{14}{137}$, from (iv) and (v), we get: $x^* = -\frac{369}{137}, z^* = \frac{29}{137}$.

The Hessian matrix is:

$$H = \begin{pmatrix} 2 & 6 & 0 \\ 6 & 2 & -3 \\ 0 & -3 & 8 \end{pmatrix}$$

it is straightforward to check that H is indefinite, so $(-\frac{369}{137}, -\frac{14}{137}, \frac{29}{137})$ is a saddle point.

<u>Problem 5</u>. A firm's production function is given by

$$Q = 2L^{1/2} + 3K^{1/2}$$

where Q, L and K denote the number of units of output, labor and capital. Labor costs are \$2 per unit, capital costs are \$1 per unit and output sells at \$8 per unit. Find the maximum profit and the values of L and K at which it is achieved.

Solution : The profit is:

$$16L^{1/2} + 24K^{1/2} - 2L - K,$$

the first order conditions are

$$(i) 8L^{-1/2} - 2 = 0, \quad (ii) 12K^{-1/2} = 1,$$

so the critical point is: $(L^*, K^*) = (16, 144)$. Now, we check the second order condition, the Hessian matrix of the profit function is:

$$H = \begin{pmatrix} -4L^{-3/2} & 0 \\ 0 & -6K^{-3/2} \end{pmatrix}$$

it is easy to see that H is negative definite, so the profit function is concave and the critical point $(16, 144)$ is a maximum. The maximum profit is: $16 \times 4 + 24 \times 12 - 2 \times 16 - 144 = 176$.