FIN 451 Exam Answers, November 8, 2007

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This is a closed-book examination. Answer all questions as directed. Mark your answers directly on the examination. On the valuation question, make sure your answer is clearly indicated. There are no trick questions on the exam. Good luck!

A. Binomial Option Pricing in One Period *40 points*

Riskless bond (interest rate is 10%):

$$100 \longrightarrow 110$$

Stock:

$$\begin{tikzcd}[row sep=small] & 190 \\ 100 \arrow[ur] \arrow[dr] & \\ & 70 \end{tikzcd}$$

European call with a strike price of 124:

$$\begin{tikzcd}[row sep=small] & ? \\ ? \arrow[ur] \arrow[dr] & \\ & ? \end{tikzcd}$$

Actual probabilities: 1/2 and 1/2.

1. What are the payoffs of the European call in the up and down states?

$$\begin{tikzcd}[row sep=small] & 66 \leftarrow \text{up state} \\ {} \arrow[ur] \arrow[dr] & \\ & 0 \leftarrow \text{down state} \end{tikzcd}$$

2. What are the risk-neutral probabilities for the two states tomorrow?

$r = 1.1$
$u = 190/100 = 1.9$
$d = 70/100 = 0.7$

$$\begin{aligned}
\pi_u &= \frac{r-d}{u-d} \\
&= \frac{1.1-0.7}{1.9-0.7} \\
&= \frac{0.4}{1.2} \\
&= 1/3
\end{aligned}$$

$\pi_d = 1 - \pi_u = 2/3$
3. What is the price of the European call today?

$\frac{1}{1.1}\left(\frac{1}{3}66 + \frac{2}{3}0\right) = 20$

4. What is the portfolio of stocks and bonds that replicates the call?

in currency units (e.g. dollars not shares)

$$\begin{array}{lcr}
1.9n_s + 1.1n_b & = & 66 \\
0.7n_s + 1.1n_b & = & 0 \\
\hline
1.2n_s & = & 66
\end{array}$$

$n_s = 55$
$n_b = \frac{-0.7n_s}{1.1} = -35$

check: $55 - 35 = 20$

B. Concepts (multiple choice) *20 points*

1. If we buy a call option, we are

a. long the stock and short volatility.
$\boxed{\text{b. long the stock and long volatility.}}$
c. short the stock and short volatility.
d. short the stock and long volatility.

2. If we sell a put option, we are

$\boxed{\text{a. long the stock and short volatility.}}$
b. long the stock and long volatility.
c. short the stock and short volatility.
d. short the stock and long volatility.

3. Which of the following is the best description of the payoff (per dollar of underlying) from holding stock index futures?

a. return on the underlying portfolio
$\boxed{\text{b. return on the underlying portfolio less the riskfree rate}}$
c. return on the underlying portfolio less the riskfree rate, plus dividends
d. return on the underlying portfolio less the riskfree rate, less dividends

4. (put-call parity) For European options, buying a call is equivalent to

a. selling a put, selling the stock, and buying the bond
b. buying a put, selling the stock, and buying the bond
c. selling a put, buying the stock, and selling the bond
$\boxed{\text{d. buying a put, buying the stock, and selling the bond}}$

5. In the Orange County financial crisis, huge amounts of money were lost betting on

a. gold prices
b. orange juice futures
$\boxed{\text{c. interest rates}}$
d. the stock market

C. Binomial Futures Option Pricing *40 points*

Two periods from now, an index futures contract matures and we model its price using a binomial model that takes on a value of \$150, \$100, or \$50:

$$
\begin{tikzcd}
 & & 150 \\
 & ? \arrow[ur] \arrow[dr] & \\
? \arrow[ur] \arrow[dr] & & 100 \\
 & ? \arrow[ur] \arrow[dr] & \\
 & & 50
\end{tikzcd}
$$

The short riskless interest rate is 25% per period. The risk-neutral probabilities of up and down moves are equal at 1/2, while the actual probability of an up move is 3/5 and the actual probability of a down move is 2/5.

Consider European and American futures call options with a strike price of \$75 and maturity two periods from now.

1. What are futures prices at each node in the tree?

$$
\begin{tikzcd}
 & & 150 \\
 & 125 \arrow[ur] \arrow[dr] & \\
100 \arrow[ur] \arrow[dr] & & 100 \\
 & 75 \arrow[ur] \arrow[dr] & \\
 & & 50
\end{tikzcd}
$$

The futures price is the expected value in the risk-neutral probabilities.

2. What are the European call option values at each node?

$$
\begin{tikzcd}
 & & 75 \\
 & 40 \arrow[ur] \arrow[dr] & \\
20 \arrow[ur] \arrow[dr] & & 25 \\
 & 10 \arrow[ur] \arrow[dr] & \\
 & & 0
\end{tikzcd}
$$

$\frac{1}{1.25}\left(\frac{1}{2}75 + \frac{1}{2}25\right) = 40$
$\frac{1}{1.25}\left(\frac{1}{2}25 + \frac{1}{2}0\right) = 10$
$\frac{1}{1.25}\left(\frac{1}{2}40 + \frac{1}{2}10\right) = 20$

3. What are the American call values at each node?

$$
\begin{tikzcd}[row sep=small]
 & & 75 \\
 & 50 \arrow[ur] \arrow[dr] & \\
25 \arrow[ur] \arrow[dr] & & 25 \\
 & 10 \arrow[ur] \arrow[dr] & \\
 & & 0
\end{tikzcd}
$$

$\max\left(125 - 75, \frac{1}{1.25}\left(\frac{1}{2}75 + \frac{1}{2}25\right)\right) = \max(50, 40) = 50$
$\max\left(75 - 75, \frac{1}{1.25}\left(\frac{1}{2}25 + \frac{1}{2}0\right)\right) = \max(0, 10) = 10$
$\max\left(100 - 75, \frac{1}{1.25}\left(\frac{1}{2}50 + \frac{1}{2}10\right)\right) = \max(25, 24) = 25$

4. (conceptual question/short answer) In this problem the risk-neutral probabilities are not equal to the actual probabilities. How can that make sense?

The stock has a positive risk premium.

D. Bonus question *20 points* (hard)

Price a claim which pays the square root of the final stock price N periods from now. The initial stock price is S_0, and in each period the stock price goes up by a factor u with risk-neutral probability 1/2 or down by a factor d, also with risk-neutral probability 1/2. One plus the riskfree rate is r at all nodes.

The final value of the claim at time N is $V_N = \sqrt{S_N}$. Over the last period, the stock price movement is

$$
\begin{tikzcd}[row sep=small]
 & uS_{N-1} \\
S_{N-1} \arrow[ur] \arrow[dr] & \\
 & dS_{N-1}
\end{tikzcd}
$$

so we can compute the value back one period as

$$
\begin{aligned}
V_{N-1} &= \frac{1}{r}\left(\frac{1}{2}\sqrt{uS_{N-1}} + \frac{1}{2}\sqrt{dS_{N-1}}\right) \\
&= \frac{1}{r}\left(\frac{1}{2}\sqrt{u} + \frac{1}{2}\sqrt{d}\right)\sqrt{S_{N-1}}
\end{aligned}
$$

Stepping back each period always has the same effect: changing the stock price to one period earlier and multiplying by the constant $\frac{1}{r}\left(\frac{1}{2}\sqrt{u}+\frac{1}{2}\sqrt{d}\right)$. Therefore,

$V_n = \frac{1}{r^{N-n}}\left(\frac{1}{2}\sqrt{u}+\frac{1}{2}\sqrt{d}\right)^{N-n}\sqrt{S_n}$

and in particular

$V_0 = \frac{1}{r^{N}}\left(\frac{1}{2}\sqrt{u}+\frac{1}{2}\sqrt{d}\right)^{N}\sqrt{S_0}$.

alternative more formal derivation:

$$\begin{aligned}
V_0 &= \frac{1}{r^N}E^*[V_N] \\
&= \frac{1}{r^N}E^*[\sqrt{S_N}] \\
&= \frac{1}{r^N}E^*\left[\sqrt{S_0\prod_{n=1}^{N}(S_n/S_{n-1})}\right] \\
&= \frac{1}{r^N}\sqrt{S_0}\prod_{n=1}^{N}E^*\left[\sqrt{S_n/S_{n-1}}\right] \\
&= \frac{1}{r^N}\sqrt{S_0}\prod_{n=1}^{N}\left(\frac{1}{2}\sqrt{u}+\frac{1}{2}\sqrt{d}\right) \\
&= \frac{1}{r^N}\left(\frac{1}{2}\sqrt{u}+\frac{1}{2}\sqrt{d}\right)^N\sqrt{S_0}
\end{aligned}$$