Volatility Smiles and Yield Curves (Author: Peter Carr)

and its financial market application

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Agenda

1 Abstract

- A simplistic benchmark model for pricing zero coupon bonds and European options with constant rate and variance.
- And then we propose a new market model, in which implied rates are assumed to be stochastic.
- Nest, we determine an entire yield curve and an entire volatility smile.
- We will also provide mathematical and financial explanations for the similarities and differences between these results.
- My opinions and additional analysis will also be covered.

1 Abstract

- Special Notes:
  - Data Source:
    - Bloomberg;
    - St. Louis FED;
  - I am especially grateful to:
    - *Prof Philip H. Dybvig*, Washington University in St. Louis, for all his help;
    - *Prof Peter Carr* (Author), New York University, for further details discussed;
    - *Dr Thao Vuong*, Washington University in St. Louis, for the presentation suggestions.
  - ***They are not responsible for any errors.***

Agenda

2 Overview

- Let's illustrate the relationships between EO and Volatility Smile; and ZCB and Yield Curve.

European Options

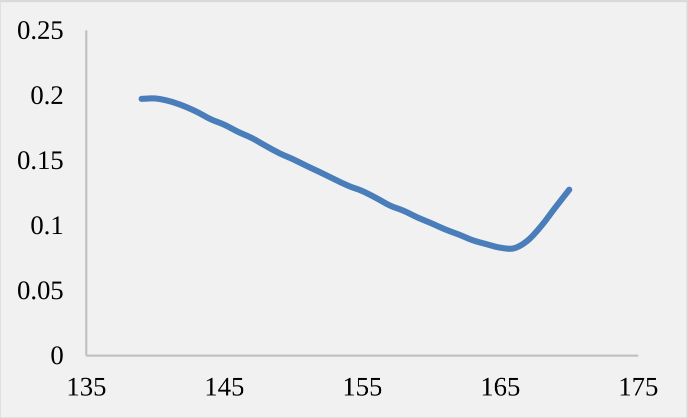
- A simple benchmark model for pricing European options can be used to define the concept of Implied Volatility (IV) by Moneyness, which can also be further used in more complex models.

Zero Coupon Bonds

- A simple benchmark model for pricing zero coupon bonds (ZCB) can be used to define the concept of Yield to Maturity (YTM), which can be further used in more complex models.

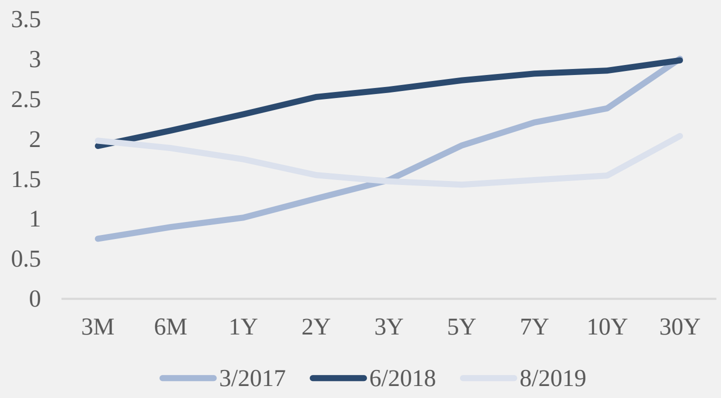
Volatility Smile (Moneyness vs Vol)

- Implied volatilities are plotted against moneyness measure.



Yield Curve (Maturity vs Yield %)

- Bond yields are plotted against the bond's term.



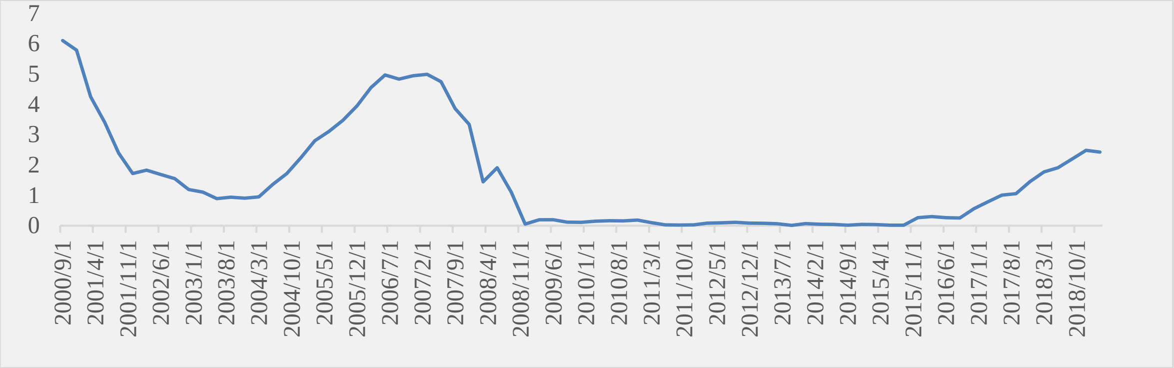
2 Overview

- Some additional parts:

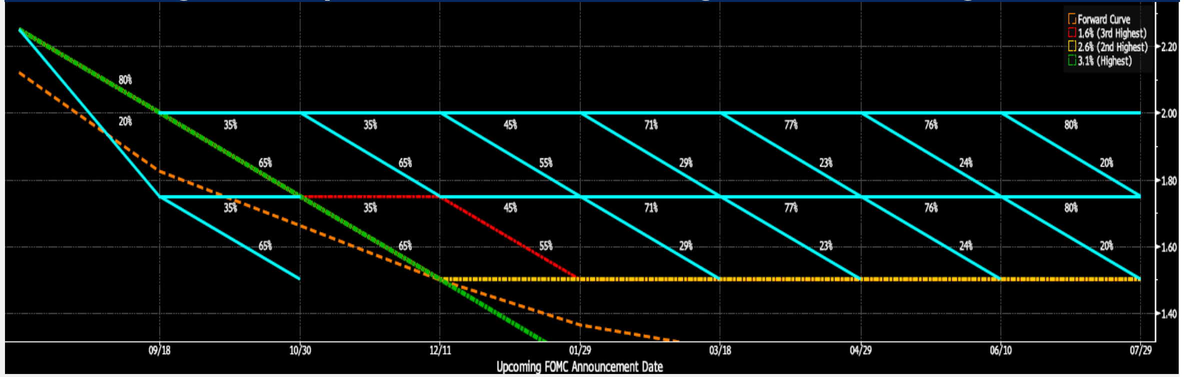
  - Academic part, Implied Volatility:
    - The assumptions of Black Scholes (1973) model.
    - How to circumvent these assumptions in practice.

  - Macroeconomics part, Yield Curve:
    - The current inverted yield curve phenomenon and the fear of recession.
    - The end of rate-rising cycle since Dec 2015.

US Federal target fund rate %



Target rate prediction in incoming FOMC meeting date



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3 Benchmark Model

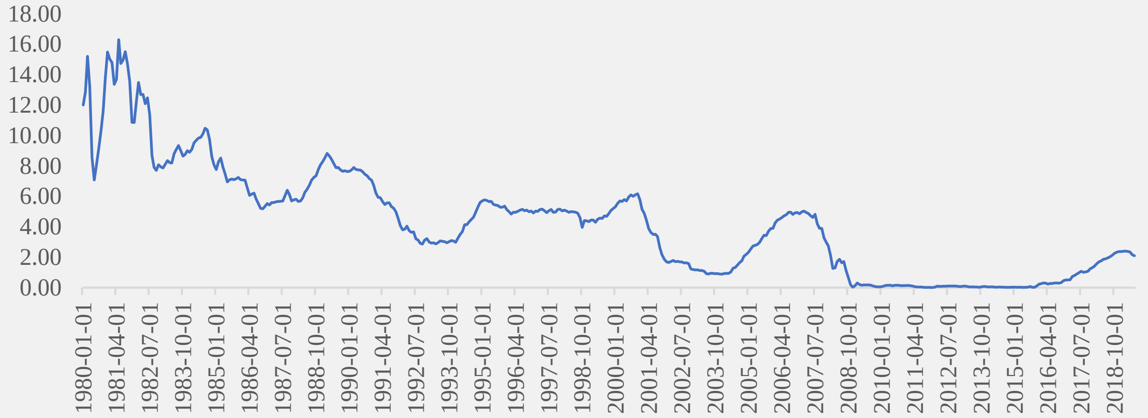
- In the benchmark model for pricing zero coupon bonds, we assume the short interest rate is constant:

$$r_t = r,$$

- in which we allow r to be any real number.

- In United States, generally we take 3-Month Treasury Bills yield as the short term interest rate.

US 3-Month Treasury Yield



3-Month Treasury Bills yield

- The 3-Month Treasury yield is the yield received for investing in a government issued treasury security that has a maturity of 3 months.

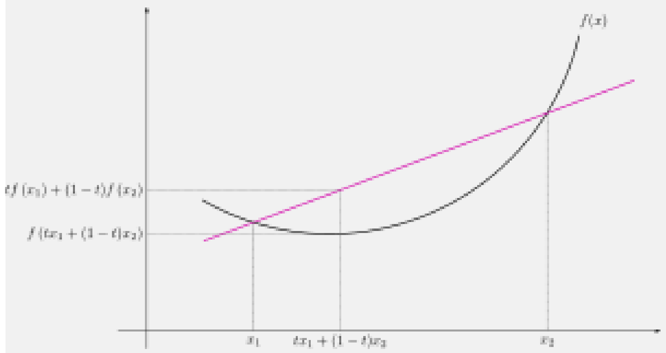
3 Benchmark Model

- In the benchmark model of a constant short interest rate r, the zero coupon bond pricing formula is given by:

$$B^c(r,\tau) = e^{-r\tau},$$

- in which r is the short-term rate, and τ is the time-to-maturity.
- The function B^c is positive, a decreasing function in r, and also strictly convex in r.

Convex Function



$$\frac{f(x_1)+f(x_2)}{2} > f\left(\frac{x_1+x_2}{2}\right)$$

Why decreasing function?

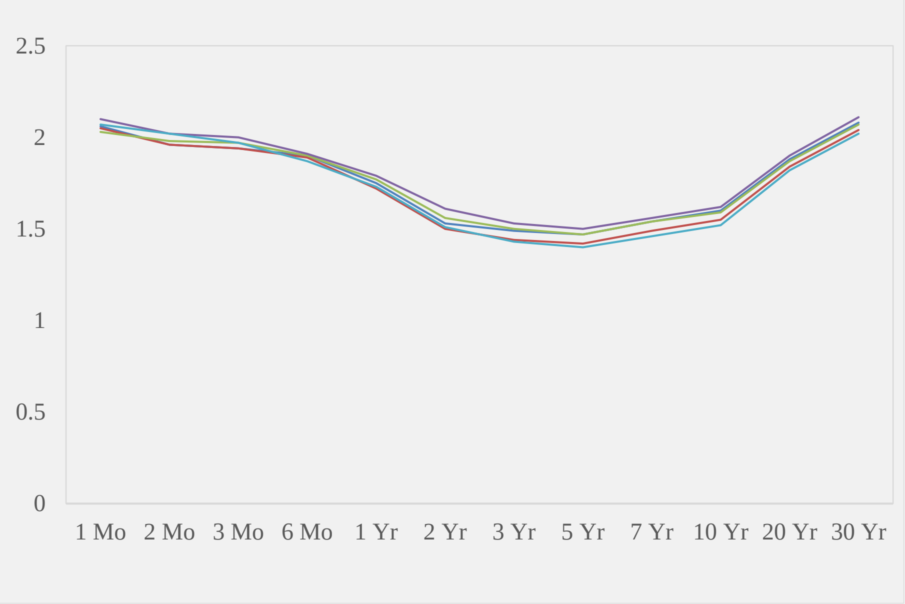
- People tends to deposit more money under higher interest rate, thus demand for bonds will be lower, and bonds price will be lower.

3 Benchmark Model

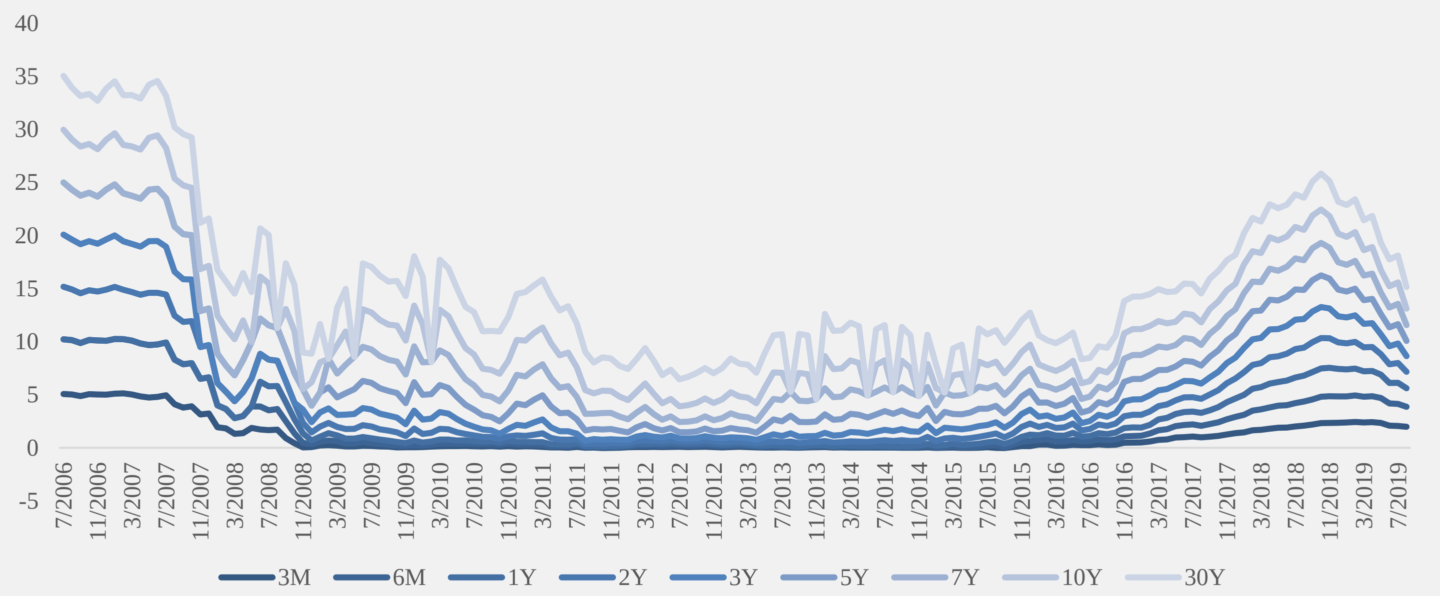
- Let $B_T(t) = B^c(r, (T-t)) = e^{-r(T-t)}$ be the price at time $t = 0$ of a zero coupon bond, which pays one dollar at its fixed maturity date $T \geq t$.

- On average, yields have been a ***concave*** function of term $(T-t)$. However, it was not always the case. So we need an innovative model that does ***not predict a flat yield curve***.

Term structure for latest five trading days



US Treasury yield under different maturities %



3 Benchmark Model

- The logarithmic derivative of the bond pricing formula $B^c\left(r,(T-t)\right)=e^{-r(T-t)}$ with respect to the time t is:

$$\frac{\partial}{\partial t} lnB^c\left(r,(T-t)\right)=r,$$

$$\frac{\frac{\partial}{\partial t}B^c\left(r,(T-t)\right)|_{r=y_t(T)}}{b_t(T)}=y_t(T),$$

- for $T \;\geq\; t \geq 0.$

- Hence, YTM is also the ***Theta*** of a 1$ investment in bonds.

> ***Theta Θ*** measures the sensitivity of the value of the derivative to the passage of time.

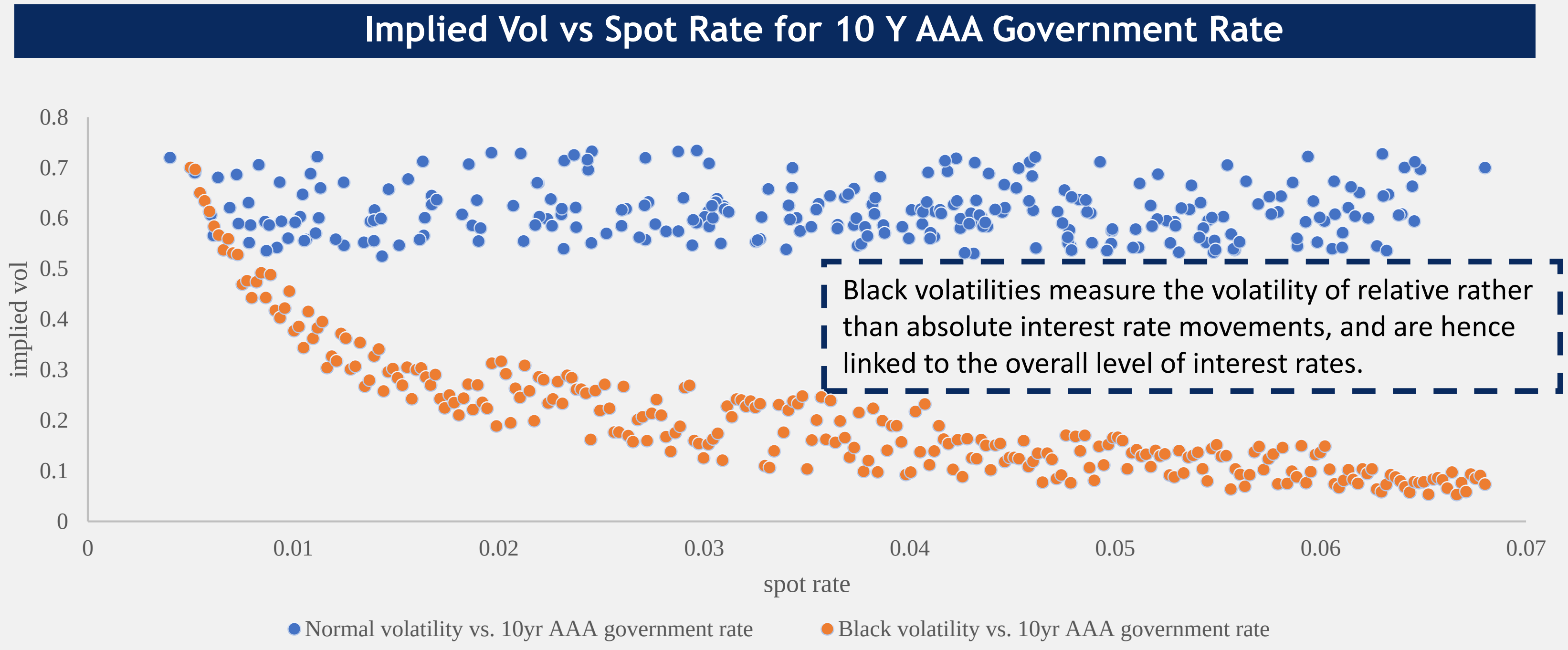
- YTM is the time component when attributing the *P&L* from investing $1 in a bond and then selling immediately afterwards.

- Time is money! (When interest rates are positive)

3 Benchmark Model

- Normal or Log-Normal?

- We can see that the market's view of the normal quote does not exhibit any pronounced dependency on the underlying interest rate level, the Black volatilities tend to increase once the underlying interest rate level is decreasing.

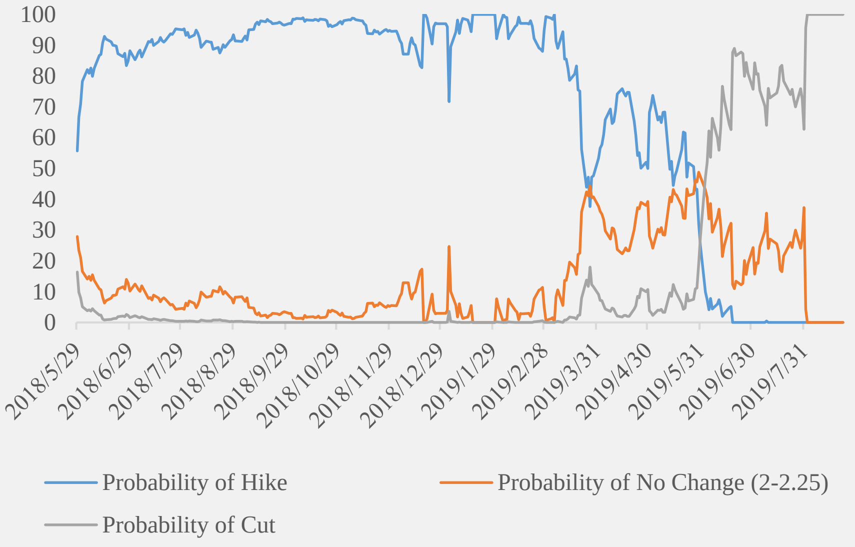


3 Benchmark Model

- Till now, we can conclude that, in the current extremely low interest rate environment, the use of normal volatility for the fixed income is a more robust approach.

- Some points about yield curves:
  - ***Yield curve shape*** reflects the market's expectations of future rate change.
  - ***Yield curve shape*** reflects the bond risk premium
  - ***Yield curve shape*** reflects the convexity benefit of bonds of different tenors.

Implied probability for rate change%



3 Benchmark Model

- In Black-Scholes (1973) option pricing model, there are some main assumptions, and even though some are impractical, BS model has been increasingly improved by circumventing these assumptions:

  - ***No transaction costs/taxes:*** A small term can be added on the BS equation.

  - ***Constant volatility/risk-free rate:*** We can use average volatility, or incorporate the stochastics.

  - ***No arbitrage:*** The most practical assumption due to the high-frequency trading. But not always.

  - ***No jumps:*** Jumps breaks the no-arbitrage, and we can divide option into two parts.

3 Benchmark Model

- For the volatility measure, we use a normal implied variance rate arising from the Bachelier model, rather than a lognormal implied variance rate arising from the BS model.

- Assumptions include:

  - Zero interest rates.

  - Spot price S of the call's underlying asset has a positive short term variance rate as constant through all the time at $a^2 > 0$.

- Thus, we have:

$$S_t = S_0 + aW_t,$$

- in which $t \geq 0$ and W is a standard Brownian Motion.

3 Benchmark Model

- What is the relationship between BS model and Bachelier model?

- We can approximate Normal (Bachelier) vols σ_N from Black vols σ_B by (actually there is a second order effect related to the product of the square of the black vol and the maturity but we can ignore here):

$$\sigma_N = \sigma_B F\sqrt{k},$$

- in which F is the forward price and k is % in moneyness.

- ***σ_B is in percentage terms, and σ_N is in dollar terms.***

Agenda

- We assume that the underlying spot price S with a constant variance a^2, but now to the market model we randomize it as a stochastic process a_t

$$S_t = S_0 + aW_t$$
$$\Longrightarrow dS_t = a_t dW_t,$$

- in which $t \geq 0$ and W_t is a standard Brownian motion.

- We assume that the implied volatility curve moves continuously and that each IV experiences the same proportional shifts:

$$d\eta_t(K) = \omega_t \eta_t(K) dZ_t,$$

- in which $t \geq 0$ and Z_t is a standard Brownian motion under $\mathbb{Q}$ measure. ω_t and η_t are bounded stochastic process.

- We define $\rho_t \in [-1, 1]$ be the correlation between W_t and Z_t, so we have

$$dW_t dZ_t = \rho_t dt,$$

- Thus the covariance between S_t and $\ln(\eta_t(K))$ can be solved as

$$\gamma_t = a_t \rho_t \omega_t,$$

- By the definition of implied volatility, $c_t(K) = C^b\big(X_t, \eta_t(K), (T-t)\big)$, in which $c_t(K)$ is the market price of the call and $C^b(X, \eta, \tau) = \eta\sqrt{\tau}N'\left(\frac{X}{\eta\sqrt{\tau}}\right) + XN\left(\frac{X}{\eta\sqrt{\tau}}\right)$. Apply Ito's formula we have

$$[\frac{\alpha_t^2}{2} C_{XX} + \gamma_t \eta_t(K) C_{\eta X} + \frac{\omega_t^2}{2} \eta_t^2 C_{\eta\eta} + C_t] C^b\big(X_t, \eta_t(K), (T-t)\big) = 0,$$

$$\Rightarrow \frac{\eta_t^2(K)}{2} = \frac{\alpha_t^2}{2} + \gamma_t (K - S_t) + \omega_t^2 \frac{(K - S_t)^2}{2} .$$

$$\frac{\eta_t^2(K)}{2} = \frac{\alpha_t^2}{2} + \gamma_t(K - S_t) + \omega_t^2 \frac{(K - S_t)^2}{2} .$$

- We notice that $\frac{\eta_t^2(K)}{2}$ is linear in α_t^2, γ_t, and ω_t^2. As a result, the entire yield curve is uniquely determined by these three parameters. Note that $\frac{\eta_t^2(K)}{2}$ can also be treated as the rate of time decay in units of gamma

$$\frac{\partial}{\partial t} C^b\big(X_t, \eta_t(K), (T - t)\big) = \frac{-\eta_t^2(K)}{2} \Gamma(X_t, \eta_t(K), (T - t)).$$

- Now we have $dS_t dS_t = \alpha_t^2 dt$, $dS_t dln(\eta_t(K)) = \gamma_t dt$, $dln(\eta_t(K)) dln(\eta_t(K)) = \omega_t^2 dt$. As a result, the implied variance rate at K is a variance as

$$\eta_t^2(K)dt = Var_t^Q\big(dS_t + (K - S)d\ln(\eta_t(K))\big)|_{S=S_t}.$$

4 Market Model

- We assume that the market gives us initial yields of zero coupon bonds at a finite number of maturities. And our objective is to connect the discrete dots, so as to produce a full yield curve.

- We assume no arbitrage and that $\mathbb{P}$ is the real world probability measure. Let $\mathbb{Q}$ be the martingale measure equivalent to $\mathbb{P}$, which arises when the money market account $e^{\int_0^t r_s ds}$ is taken as the numeraire.

- Suppose that under $\mathbb{Q}$ measure, the yield curve evolves continuously and only by parallel shifts

$$dy_t = \delta_t dt + v_t dZ_t,$$

- in which $t \geq 0$ and Z_t is a standard Brownian motion under $\mathbb{Q}$ measure.

4 Market Model

- Let $b_t(T)$ be the market price of a bond. By the definition of yield to maturity $y_t(\tau)$, we have

$$b_t(T) = B^c(y_t(T), T-t).$$

- Then we consider Ito's formula, to let the process $e^{\int_0^t r_s ds} B^c(y_t(T), T-t)$ be a **martingale**,

$$E_t^Q e^{\int_0^t r_s ds} B^c(y_t(T), T-t) = [\frac{v_t^2}{2} C_{yy} + \delta_t C_y - r_t + C_t] C^b\big(X_t, \eta_t(K), (T-t)\big) = 0,$$

$$\Longrightarrow y_t(T) = r_t + \delta_t(T-t) - v_t^2 \frac{(T-t)^2}{2},$$

- by substituting

$$\frac{\partial}{\partial t} B^c\big(y_t(T), (T-t)\big) = y_t(T) B^c(y_t(T), T-t).$$

$$\frac{\partial}{\partial y} B^c\big(y_t(T), (T-t)\big) = -(T-t) B^c(y_t(T), T-t).$$

$$\frac{\partial}{\partial y^2} B^c\big(y_t(T), (T-t)\big) = (T-t)^2 B^c(y_t(T), T-t).$$

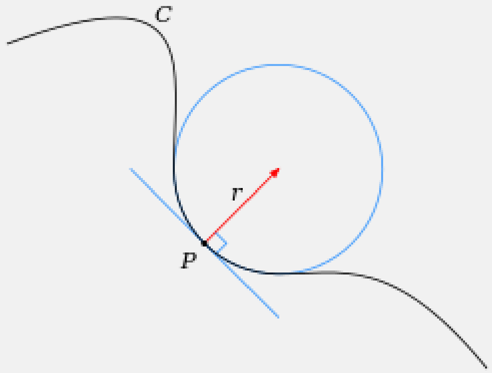
- What is the curvature?

- Some applications in financial mathematics:

  - Taylor Series: $F(x) = F(a) + F'(a)(x-a) + \frac{F''(a)}{2!}(x-a)^2 + \cdots$

  - Gamma (Greeks): $\Gamma = \frac{\partial^2 V}{\partial S^2}$

  - Convexity (Bond): $C = \frac{1}{B}\frac{\partial^2 (B)}{\partial r^2}$



Curvature in the plane curve

4 Market Model

$$y_t(T) = r_t + \delta_t(T-t) - v_t^2 \frac{(T-t)^2}{2}.$$

- We notice that the entire yield curve $y_t(T)$ is uniquely determined by these three parameters in r_t, δ_t, and v_t^2.
- And we can see arbitrage-free implied variance smile experiences the same proportional shifts

$$\frac{\eta_t^2(K)}{2} = \frac{\alpha_t^2}{2} + \gamma_t(K - S_t) + \omega_t^2 \frac{(K-S_t)^2}{2}.$$

- Both curves have three components:
  - For yield, the intercept is the short rate r_t, the slope is δ_t, and the curvature is $-v_t^2$.
  - For IV, the intercept is the short variance rate $\frac{\alpha_t^2}{2}$, the slope is γ_t, and the curvature is ω_t^2.

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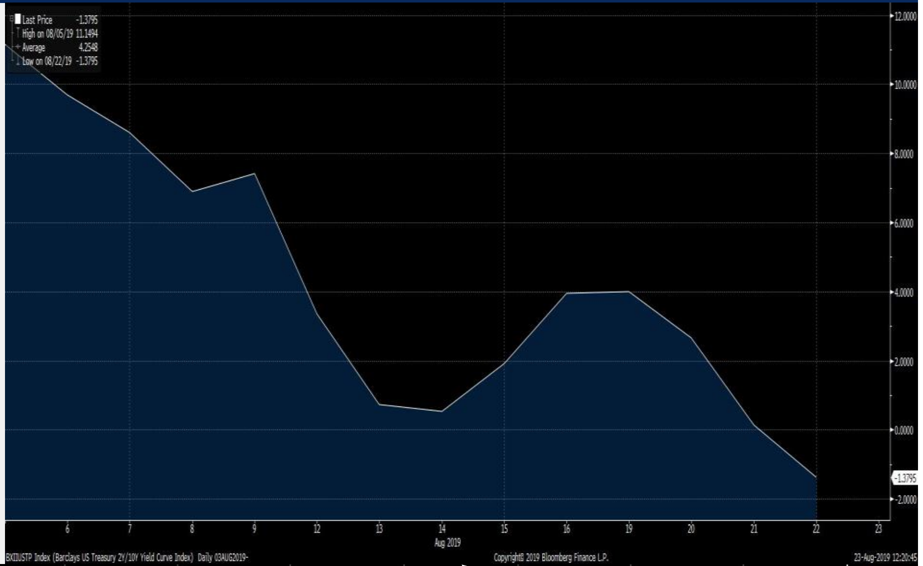
5 Macroeconomics

Barclays US Treasury 2Y/10Y Yield Curve Index

- The (long-term) negative spread between 10-year Treasury and 2-year Treasury indicates the (more possible) recession.

- Negative spread means ration is less than 1. (Barclays)

- It was the case (temporarily) in August, but the first time since 2007. Now the spread is (very) close to 0.



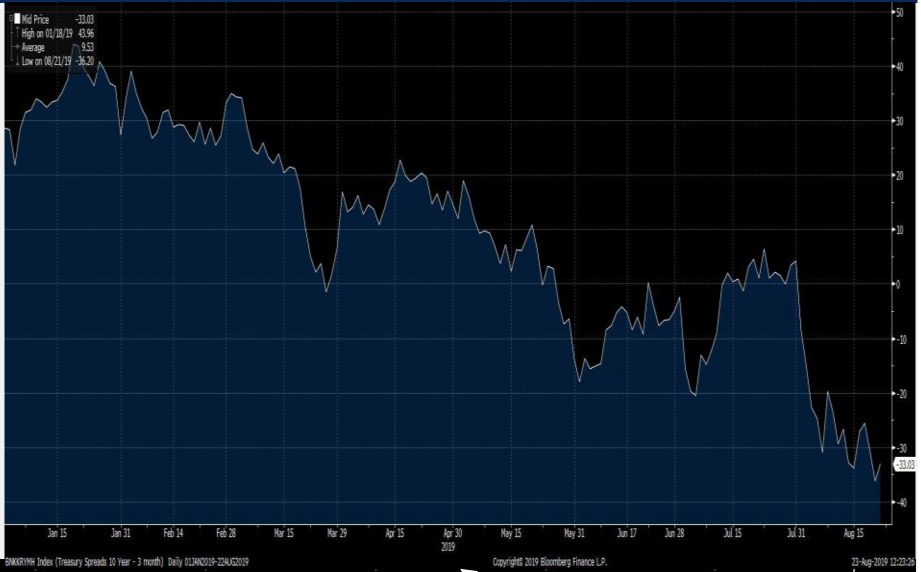
5 Macroeconomics

Treasury Spreads 10Y - 3M

- The (long-term) negative spread between 10-year Treasury and 3-month Treasury indicates the recession.

- Normal yield curve should be concave, as people need more compensation for less liquidity.

- It was the case (temporarily) in March, and (much lower) after July.



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6 Conclusion

- ***Variance rates play a similar role in option pricing as interest rates do in bond pricing.***

- Market models are used to develop ***quadratic*** arbitrage-free curves in both cases.

- For curve construction, the model can only be used to value ***bonds*** or ***path-independent options***, or the linear combinations of both coupon bonds and payoffs.

- ***In the current extremely low interest rate environment, the use of normal volatility for fixed income produces is a more robust approach.***

- Implied volatility quotes remove the effect of the parameters not related to volatility, and hence allow for a comparison of swaption prices among different swaptions and over different time.

6 Conclusion

- The stochastic yield model says that if yields move only by ***parallel*** shifts, then the ***absence of arbitrage*** forces the yield curve at time 0 to ***be quadratic*** in term opening down.

- The model allows this. If we are given 3 yields at maturities $0 \leq T_1 < T_2 < T_3$ with yields $y_3 < y_2 < y_1$, then the model allows this inverted yield curve as well, as long as the 3 yields plot concave in maturity.

- If market yield curve is observed to not be negative quadratic, there are 3 possibilities:

  - The parallel shift yield dynamics are right and there is arbitrage (form a butterfly spread).

  - The parallel shift yield dynamics are wrong and there is no arbitrage.

  - The parallel shift yield dynamics are wrong and there is arbitrage.